Exercise 10

Show that $\text{conv}(S)$ is convex, $S \subseteq \mathbb{R}^n$

Solution: $\text{conv}(S)$ consists of all points on the form $\sum_{j=1}^{n} \lambda_j x_j$, $x_j \in S$, $\sum \lambda_i = 1$.

We need to show: $(1-\lambda)\sum_{i=1}^{n} \lambda_i x_i + \lambda \sum_{k=1}^{m} \mu_k y_k \in \text{conv}(S)$ $\left(\sum_{i=1}^{n} \lambda_i = \sum_{k=1}^{m} \mu_k = 1\right)$

coefficients sum to

$$\sum_{i=1}^{n} (1-\lambda)\lambda_j + \sum_{k=1}^{m} \lambda \mu_k$$

$$= (1-\lambda)\sum_{i=1}^{n} \lambda_j + \lambda \sum_{k=1}^{m} \mu_k$$

$$= (1-\lambda)\cdot 1 + \lambda \cdot 1 = 1-\lambda+\lambda = 1$$

Exercise 11

$S = \{0, 2\}$ $\quad$ $T = \{0, 1, 2\}$

$\text{conv}(S) = [0, 2]$ $\quad$ $\text{conv}(T) = [0, 2]$

Exercise 12

Assume $S \subseteq T$

$\text{conv}(S)$ = all points on the form $\sum_{i=1}^{n} \lambda_i x_i$, $x_i \in S$

$\text{conv}(T)$ = all points on the form $\sum_{j=1}^{m} \lambda_j x_j$, $x_i \in T$ $\quad$ $\cup$

$\Downarrow$

Exercise 13

Show that $\text{conv}(S) = S$ when S is convex.

Solution: We know $S \subseteq \text{conv}(S)$ (choose $x^1 = x^2 = x$ $\quad$ $(1-\lambda)x_1 + \lambda x_2$

$\text{conv}(S) \subseteq S$:

$\sum \lambda_i x_i$ $x_i \in S$, $\sum \lambda_i = 1$

We have: $\sum_{i=1}^{n} \lambda_i x_i = \lambda_1 x_1 + (1-\lambda_1)\sum_{i=2}^{n} \frac{\lambda_i}{1-\lambda_1} x_i$

sum to $\frac{\lambda_2+\lambda_3+\cdots+\lambda_n}{1-\lambda_1}$ $\quad$ $\frac{1-\lambda_1}{1-\lambda_1} = 1$

conv. comb. of n-1 points, assumed by induction to be in S

so that $\sum_{i=1}^{n} \lambda_i x_i \in S$ also, since this is a convex comb. of points in S.

Exercise 14 What is $conv(S)$ for $S = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$?

Solution: For $x, y \in S$:

$$\|(1-\lambda)x + \lambda y\|_2 \leq (1-\lambda)\|x\|_2 + \lambda\|y\|_2 = 1-\lambda+\lambda = 1$$

$$\Rightarrow conv(S) \subseteq B(0,1)$$

Proof that: $B(0,1) \subseteq conv(S)$:

We have that $0 \in conv(S)$: $\left((0,0) = \frac{1}{2}(1,0) + \frac{1}{2}(-1,0)\right)$

if $x \neq 0$, $\|x\|_2 \leq 1$, then x lies on the line segment from $\frac{x}{\|x\|_2}$ to $\frac{-x}{\|x\|_2}$ $\Rightarrow$ $x \in conv(S)$

Therefore: $conv(S) \supseteq B(0,1)$

$$\rotatebox{90}{$\subseteq$}$$

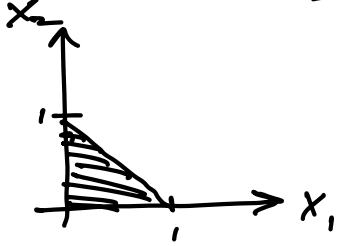
$$conv(S) = B(0,1)$$

Exercise 15 $S = \{(0,0), (1,0), (0,1)\}$

$$conv(S) = \{\lambda_1(0,0) + \lambda_2(1,0) + \lambda_3(0,1) : \lambda_1+\lambda_2+\lambda_3 = 1, \lambda_i \geq 0\}$$

$$= \{(\lambda_2, \lambda_3) : \lambda_2+\lambda_3 \leq 1, \lambda_2, \lambda_3 \geq 0\}$$

$$= \{(x_1, x_2) : x_1+x_2 \leq 1, x_1, x_2 \geq 0\}$$



Exercise 16 $S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}$

Two ways to prove that $conv(S) = \{x : 0 \leq x_i \leq 1 \ \forall i\}$

1. Extreme points of $\{x : 0 \leq x_i \leq 1\}$ must clearly satisfy $x_i = 0$ or 1 for all i $\Rightarrow$ S are its extreme points:

$$\{x : 0 \leq x_i \leq 1\} = conv(S)$$

2. Clearly $\text{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}$ 

proof: $(x,y) = (1-x)(0,y) + x(1,y)$

$(0,y) = (1-y)(0,0) + y(0,1)$

$(1,y) = (1-y)(1,0) + y(1,1)$

Since also $(x,y,z) = (1-z)(x,y,0) + z(x,y,1)$:

$$\{x : 0 \le x_i \le 1 \ \forall i\} \subseteq \text{conv}(S)$$

Also, since any convex combination of the points in S satisfy $0 \le x_i \le 1$, we also have $\text{conv}(S) \subseteq \{x : 0 \le x_i \le 1\}$

$$\Downarrow$$

$$\text{conv}(S) = \{x : 0 \le x_i \le 1\}.$$

Consider $S \setminus \{(1,1,1)\}$ (7 points)

We have $S \setminus \{(1,1,1)\} \subseteq \{x : x_1 + x_2 + x_3 \le 2,\ 0 \le x_i \le 1\}$

I claim that

$$\text{conv}(S \setminus \{(1,1,1)\}) = \{x : x_1 + x_2 + x_3 \le 2,\ 0 \le x_i \le 1\}$$

Enough to prove that $S \setminus \{(1,1,1)\}$ are the extreme points of this set.

The extreme points are the basic feasible solutions of

$$\underbrace{x_1 + x_2 + x_3 \le 2}_{w_0} \qquad \underbrace{0 \le x_i \le 1}_{w_i}$$

There are 4 basic variables.

- If w_0 is basic, then all $x_i = 0$ or 1 (both x_i and w_i can't be nonbasic)
- If w_0 is nonbasic: 4 of $x_1, x_2, x_3, w_1, w_2, w_3$ are basic

$$\Downarrow$$

2 must be nonbasic: two of x_i must be 0 or 1.

Since $x_1+x_2+x_3=2$ the third must also be 0 or 1

So, a basic feasible solution must be in $S\setminus\{(1,1,1)\}$

$\Downarrow$

$S\setminus\{(1,1,1)\}$ are the extreme points of $\{x: x_1+x_2+x_3\le 2,\ 0\le x_i\le 1\}$.

$\Downarrow$

$\operatorname{conv}(S\setminus\{(1,1,1)\}) = \{x: x_1+x_2+x_3\le 2,\ 0\le x_i\le 1\}$.

Exercise 17

Lemma 4: $Ax\le b$ has a solution

$\Leftrightarrow$

$y^T b\ge 0$ $\forall y$ with $y^TA=0$, $y\ge 0$

$$\begin{bmatrix} Ax=b \\ x\ge 0 \end{bmatrix} \longrightarrow \begin{bmatrix} Ax\le b \\ -Ax\le -b \\ -x\le 0 \end{bmatrix} \longrightarrow \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

Lemma 4: this has a solution if and only if

$$[y_1^T\ y_2^T\ y_3^T]\begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}\ge 0 \quad \text{for all} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ with } [y_1^T y_2^T y_3^T]\begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0$$

$$y_1^Tb - y_2^Tb = (y_1-y_2)^Tb \quad \text{for all } y \text{ with } y_1^TA - y_2^TA - y_3^T = 0$$

(y unconstrained)

$$(y_1-y_2)^TA = y_3^T \ge 0$$

We get: $y^Tb\ge 0$ for all y with $y^TA\ge 0$ $\qquad y^TA\ge 0$

Exercise 18

(i) $P=\{x\in\mathbb{R}^n: a_1x_1+\cdots+a_nx_n=b,\ x_i\ge 0\}$

If two of the x_i's are >0, then x is not an extreme point.

So, only one x_i can be nonzero $\Rightarrow$ $a_ix_i=b \Rightarrow x_i=\frac{b}{a_i}$

$\Rightarrow$ extreme points are $(0,\dots,0,\frac{b}{a_i},0,\dots,0)$

Exercise 19 $P = \{x \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$.

extreme points are $x_i = a_i$ or b_i for all i

2^n possible extreme points.