Conclusion from last lecture

By removing the row corresponding to the root node, the network flow problem looks as

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \tilde{A}x = -\tilde{b} \\ & x \geq 0 \end{aligned}$$

where $\tilde{A}, \tilde{b}$ are A, b with one row removed, and wher $\tilde{A}$ has linearly independent rows.

Theorem 14.1 A square submatrix of $\tilde{A}$ gives a basis if and only if the corresponing edges form a spanning tree.

Proof: $\Uparrow$: From example above, any spanning tree gives a basis, since the corresponding matrix is triangular $\Rightarrow$ invertible

$\Downarrow$: Assume that the columns do not form a spanning tree. Then we have $|N|-1$ edges that do not visit all $|N|$ nodes. Graph theory then says that there exists a cycle. The corresponding columns then sum to 0, so that we have a linear dependence of the columns, so that the matrix is not invertible $\Rightarrow$ not a basis ∎

So: Spanning trees $\Leftrightarrow$ LP bases in $\tilde{A}$.

Also solves exercise 14.12.

Chapter 14, cont.

How we find the dual variables / dual slack variable.

$$\begin{aligned} \max \quad & -\sum_{i \in N} b_i y_i \\ \text{s.t.} \quad & y_j - y_i + z_{ij} = c_{ij} \quad (i,j) \in A \\ & z_{ij} \geq 0 \end{aligned}$$

Set $y_r = 0$ (rotnode).

if (i,j) in spanning tree : X_{ij} basic $\Rightarrow z_{ij}$ nonbasic $\Rightarrow z_{ij} = 0$.

$y_j - y_i = c_{ij}$ (all y_i can be found from this, start with root)

After the y_i have been computed, solve

$$y_j - y_i + z_{ij} = c_{ij}$$

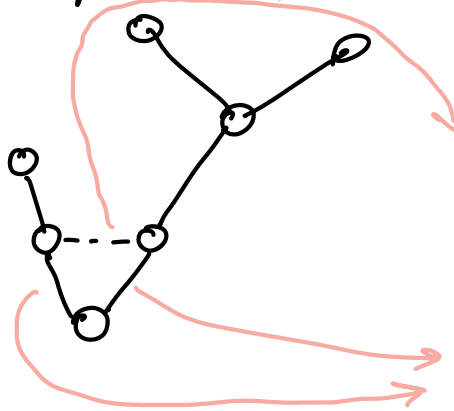
for z_{ij} for each (i,j) not in the spanning tree.

Primal network simplex metode

Start with a spanning tree with $x_B \geq 0$

1) Check optimality (i.e. $z_{ij} \geq 0$ outside spanning tree, dual feasibility).

2) If not optimal : perform a pivot.



entering variable : some edge (u,v) enters the spanning tree, creating a cycle $\Rightarrow x_{uv}$ enters basis.

leaving variable : some edge in this cycle leaves spanning tree. Two choices

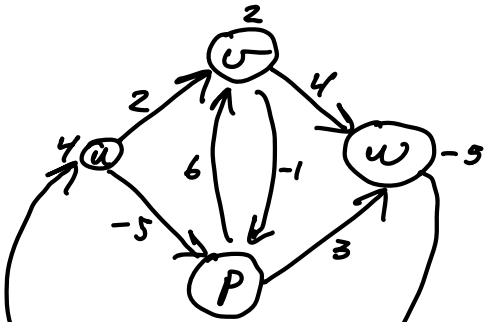
So: Find (u,v) outside spanning tree with $z_{uv} < 0$

Smallest such → largest coefficient rule.

(u,v) : will introduce a cycle.

Take another edge out of the cycle, to get back to a tree.

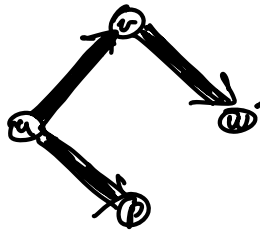
Example:



b_i : supplies shown at nodes

c_{ij} : costs, shown at edges

1. Find a spanning tree T,



→ choose as root node, $y_w = 0$

drop flow balance at w.

2. Compute tree solution at X, apply flow balance at

(p) $-x_{up} = -1 \Rightarrow \underline{x_{up} = 1}$

(u) $x_{up} + x_{uv} = 4 \Rightarrow x_{uv} = 4 - x_{up} = 4 - 1 = 3 \Rightarrow \underline{x_{uv} = 3}$

(v) $x_{vw} - x_{uv} = 2 \Rightarrow x_{vw} = x_{uv} + 2 = 3 + 2 = 5 \Rightarrow \underline{x_{vw} = 5}$

ser at $X \geq 0$!

3. Compute $y_j - y_i = c_{ij}$ for $(i,j) \in T$,

(v,w) : $y_w - y_v = c_{vw} \Rightarrow \underline{y_v = -4}$

(u,v) : $y_v - y_u = c_{uv} \Rightarrow y_u = y_v - c_{uv} = -4 - 2 = \underline{-6}$

(u,p) : $y_p - y_u = c_{up} \Rightarrow y_p = -6 - 5 = \underline{-11}$

4. Compute z_{ij} (only outside the tree, since $z_{ij} = 0$ inside tree)

(w,u) : $y_u - y_w + z_{wu} = c_{wu} \Rightarrow z_{wu} = 6 + 0 + 7 = \underline{13}$

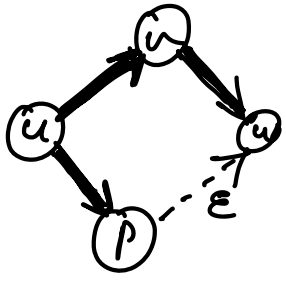
(p,w) : $y_w - y_p + z_{pw} = c_{pw} \Rightarrow z_{pw} = 0 - 11 + 3 = \underline{-8}$

(p,v) : $y_v - y_p + z_{pv} = c_{pv} \Rightarrow z_{pv} = 4 - 11 + 6 = \underline{-1}$

(v,p) : $y_p - y_v + z_{vp} = c_{vp} \Rightarrow z_{vp} = 11 - 4 - 1 = \underline{6}$

Since $z_{pw} < 0$, this is not optimal (not dual feasible)

x_{pw} enters the ~~basis~~ (max coefficient rule), creating a loop.



How much can we increase the flow at (p,u) ?

Flowbalance at (p) $x_{pw}=\varepsilon$: $\tilde{x}_{up} = x_{up}+\varepsilon = 1+\varepsilon$

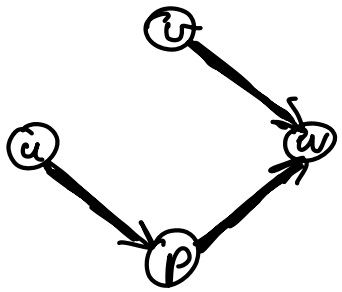
(u) $x_{pw}=\varepsilon$: $\tilde{x}_{uv} = x_{vw}-\varepsilon = 3-\varepsilon$

(v) $x_{pw}=\varepsilon$: $\tilde{x}_{vw} = x_{vw}-\varepsilon = 5-\varepsilon$

We see that we can increase ε to 3, and that x_{uv} then leaves (as $x_{uv}=0$)

New spanning tree:

$\varepsilon=3$



T_2

Updated tree solution / primal flow:

$x_{pw}=3$ (entered)
$x_{up}=4$
$x_{uv}=0$ (left)
$x_{vw}=2$

Second iteration

3. Compute y : $y_j - y_i = c_{ij}$ for $(i,j)\in T_2$:

(v,w) : $y_w - y_v = c_{vw} \Rightarrow y_v = 0-4 = \underline{-4}$

(u,p) : $y_p - y_u = c_{up} \Rightarrow y_u = y_p - c_{up} = -3+5 = \underline{2}$

(p,w) : $y_w - y_p = c_{pw} \Rightarrow \underline{y_p = -3}$ (first)

4. Compute z_{ij} for $(i,j)\notin T_2$:

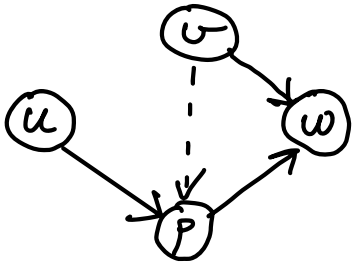
(w,u) : $y_u - y_w + z_{wu} = c_{wu} \Rightarrow 2-0+z_{wu}=7 \Rightarrow \underline{z_{wu}=5}$

(p,v) : $y_v - y_p + z_{pv} = c_{pv} \Rightarrow -4+3+z_{pv}=6 \Rightarrow z_{pv}=7$

(v,p) : $y_p - y_v + z_{vp} = c_{vp} \Rightarrow -3+4+z_{vp}=-1 \Rightarrow \underline{z_{vp}=-2}$

(u,v) : $y_v - y_u + z_{uv} = c_{uv} \Rightarrow -4-2+z_{uv}=2 \Rightarrow \underline{z_{uv}=8}$

Since $z_{vp}<0$, this is not optimal, x_{vp} enters basis, creating a cycle



Flow balance at

v: $x_{vp}=\varepsilon \Rightarrow \tilde{x}_{vw} = x_{vw} - \varepsilon = 2-\varepsilon$

p: $x_{vp}=\varepsilon \Rightarrow \tilde{x}_{pw} = x_{pw} + \varepsilon = 3+\varepsilon$

We see that we can increase ε to 2, and that x_{vw} then leaves.

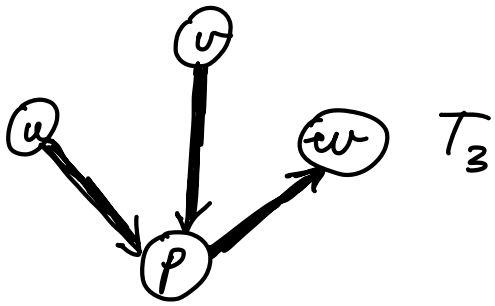
Updated tree solution:

$x_{pw} = 5$

$x_{vw} = 0$ (left)

$x_{up} = 4$ (did not change, outside cycle)

$x_{vp} = 2$ (entered)



T_3

Third iteration: Compute y_i : $y_j - y_i = c_{ij}$, $(i,j) \in T_3$:

3.

(u,p) : $y_p - y_u = c_{up} \Rightarrow -3 - y_u = -5 \Rightarrow \underline{y_u = 2}$

(p,w) : $y_w - y_p = c_{pw} \Rightarrow 0 - y_p = 3 \Rightarrow \underline{y_p = -3}$

(v,p) : $y_p - y_v = c_{vp} \Rightarrow -3 - y_v = -1 \Rightarrow \underline{y_v = -2}$

4. Compute z_{ij} for $(i,j) \notin T_3$:

(u,v) : $y_v - y_u + z_{uv} = c_{uv} \Rightarrow -2 - 2 + z_{uv} = 2 \Rightarrow z_{uv} = 6$

(v,w) : $y_w - y_v + z_{vw} = c_{vw} \Rightarrow 0 + 2 + z_{vw} = 4 \Rightarrow z_{vw} = 2$

(w,u) : $y_u - y_w + z_{wu} = c_{wu} \Rightarrow 2 - 0 + z_{wu} = 7 \Rightarrow z_{wu} = 5$

(p,v) : $y_v - y_p + z_{pv} = c_{pv} \Rightarrow -2 + 3 + z_{pv} = 6 \Rightarrow z_{pv} = 5$

This is optimal!

optimal value: $\sum_{(i,j)\in T_3} c_{ij} x_{ij} = c_{pw}x_{pw} + c_{up}x_{up} + c_{vp}x_{vp}$

$= 3 \cdot 5 - 5 \cdot 4 - 1 \cdot 2$

$= 15 - 20 - 2 = -7$