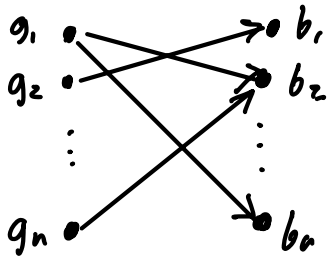
**Theorem 14.3** König's theorem (application of integer flows)

We have $n$ girls / boys, and assume that every boy/girl knows $k$ girls/boys. Then $n$ marriages can be arranged where everyone knows his/her spouse

**Proof:** Represent girls / boys as nodes $g_1, \dots, g_n$ / $b_1, \dots, b_n$

Place girls to the left, boys to the right, arrows left to right "knowing the other"




Called a bipartite graph

Set supplies: $b = \begin{cases} 1 : \text{girl nodes} \\ -1 : \text{boy nodes} \end{cases}$

This problem is feasible: Set $x_j = \frac{1}{k}$ for all edges:

flow balance at: boy nodes: $-\frac{1}{k} - \dots - \frac{1}{k} = -1$

girl nodes: $\frac{1}{k} + \dots + \frac{1}{k} = 1$

$\Rightarrow$ flow balance ok $\Rightarrow$ problem feasible $\Rightarrow$ has a basic feasible solution

Also: Any basic feasible solution to a network flow problem with integer supplies, is also integer-valued (see this from computation of tree solution).

At a girl node $x_1 + \dots + x_k = 1$ (if the girl knows boy $1, \dots, k$)

Since also $x_j \geq 0$, $x_j = 0$ for exactly one $j$, which gives us the marriage we seek. ∎

# Exercise 14.3

a) primal flows: outflow - inflow = $b_i$ (flow balance in (i))

(f) (leaf): $-x_{af} = 0 \Rightarrow \underline{x_{af} = 0}$

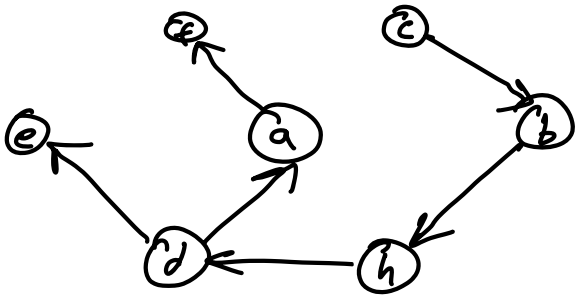
(a) $x_{af} - x_{da} = 1 \Rightarrow \underline{x_{da} = 1}$

(e) $-x_{de} = -6 \Rightarrow \underline{x_{de} = 6}$

(c) $\underline{x_{cb} = 6}$

(b) $x_{bh} - x_{cb} = 7 \Rightarrow \underline{x_{bh} = 13}$

(h) $x_{hd} - x_{bh} = 0 \Rightarrow \underline{x_{hd} = 13}$




b) Dual variable: $y_j - y_i = c_{ij}$ for $(i,j) \in T$

La e vore rotnode ($y_e = 0$)

| | | | |
|---|---|---|---|
| $(d,e)$: | $y_e - y_d = 6$ | $\Rightarrow$ | $\underline{y_d = -6}$ |
| $(d,a)$: | $y_a - y_d = 0$ | $\Rightarrow$ | $\underline{y_a = -6}$ |
| $(a,f)$: | | $\Rightarrow$ | $\underline{y_f = 0}$ |
| $(h,d)$: | | $\Rightarrow$ | $\underline{y_h = -2}$ |
| $(b,h)$: | | $\Rightarrow$ | $\underline{y_b = -16}$ |
| $(c,b)$: | | $\Rightarrow$ | $\underline{y_c = -23}$ |

c) Dual slack for non-tree arcs: $y_j - y_i + z_{ij} = c_{ij}$ for $(i,j) \notin T$

$(e,f)$ : $y_f - y_e + z_{ef} = c_{ef} \Rightarrow 0 - 0 + z_{ef} = 8 \Rightarrow z_{ef} = 8$

$(c,f)$ : $y_f - y_c + z_{cf} = c_{cf} \Rightarrow 0 + 23 + z_{cf} = 3 \Rightarrow z_{cf} = -20$

$(a,c)$ :

$(a,e)$ :

$(a,b)$ :

$(h,a)$ :

$z_{ac} = 24$

$z_{ae} = -3$

$z_{ab} = 10$

$z_{ha} = 3$

ikke optimalt!

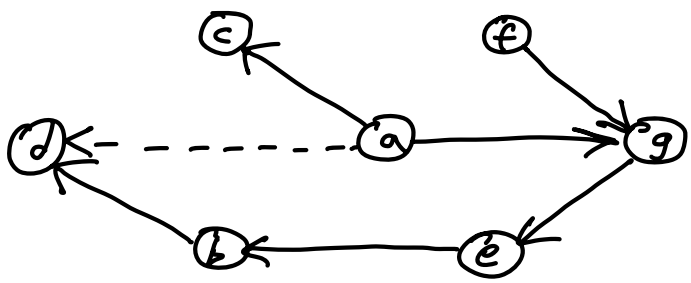
## Exercise 14.4

a) Note that $X_B \geq 0$

Largest coefficient rule : pick most negative dual slack:

Here this is $z_{ad} = -13 \Rightarrow X_{ad}$ enters.

b) Let $X_{ad}$ be increased to $\varepsilon$. Adding $(a,d)$ to the spanning tree gives cycle $(a, d, b, e, g, a)$




only flow in the cycle changes when $\varepsilon$ is increased

Flow balance within cycle:

$$\tilde{X}_{ad} = \varepsilon$$

$$\tilde{X}_{bd} = 6 - \varepsilon$$

$$\tilde{X}_{eb} = 14 - \varepsilon$$

$$\tilde{X}_{ge} = 14 - \varepsilon$$

$$\tilde{X}_{ag} = 1 - \varepsilon \longrightarrow \text{becomes } 0 \text{ first, for } \varepsilon = 1$$

$x_{ag}$ leaves

c/ New tree solution:

$x_{ad} = 1$

$x_{bd} = 5$

$x_{eb} = 13$

$x_{ge} = 13$

$x_{ga} = 0$

$x_{ac} = 0$

$x_{fg} = 6$

unchanged since they are outside cycle.