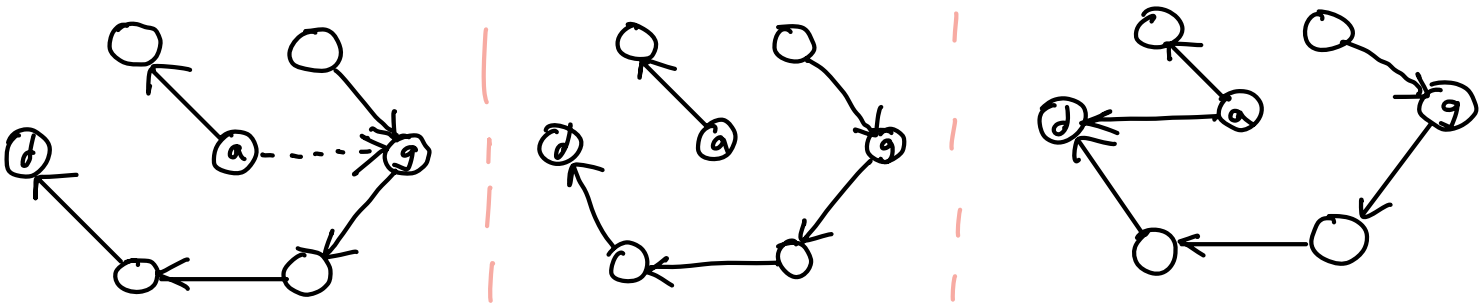
How to update the dual (slack) variables



Before (a,g) leaves (a,d) enters

One subtree contains d, another does not.
Assume that the root node is one of the nodes in the d-subtree.
Since $y_j - y_i = c_{ij}$ in this subtree, none of the y_i change in this subtree (since y_i for the root node does not change).

(a,d) enters: $y_d - y_a + z_{ad} = c_{ad}$

$\tilde{y}_d - \tilde{y}_a = c_{ad}$

subtract: $-y_a + \tilde{y}_a + z_{ad} = 0$ (since $y_d = \tilde{y}_d$)

$$\tilde{y}_a = y_a - z_{ad} = y_a + 13 \text{ ~~from book~~}$$

$\Rightarrow y_a$ is decreased with $z_{ad} = -13$.

Since $y_j - y_i + z_{ij} = c_{ij}$ for (i,j) outside spanning tree, z_{ij} is not changed when i and j are in the same subtree
Arcs (i,j) bridging the two subtrees change as follows:

(*) arcs i from d-tree to j the other tree:

$$y_j - y_i + z_{ij} = c_{ij} = \tilde{y}_j - \tilde{y}_i + \tilde{z}_{ij}$$
$$= y_j + 13 - y_i + \tilde{z}_{ij} \Rightarrow \tilde{z}_{ij} = z_{ij} - 13$$

So z_{ij} is increased with dual slack of the entering arc.

(**) arcs from the other tree to the d-tree:
One shows similarly z_{ij} is decreased with dual slack of entering arc.

From d-tree

$\tilde{z}_{dc} = 18 - 13 = 5$

$\tilde{z}_{ba} = 10 - 13 = -3$

$\tilde{z}_{ea} = 13 - 13 = 0$

$\tilde{z}_{fc} = -10 - 13 = -23$

From other tree

$\tilde{z}_{ad} = -13 + 13 = 0$

$\tilde{z}_{ag} = 0 + 13 = 13$

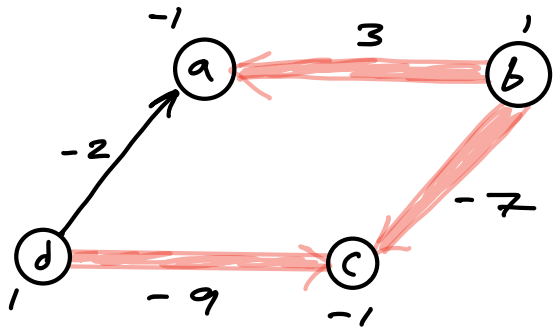
$\tilde{z}_{af} = 14 + 13 = 27$

Conclusion:

1. When only the costs and the spanning tree are known, we can compute the primal flow, the dual variables, and the dual slack variables.
2. When the spanning tree, the primal flow, and the dual slack variables are known, we can update these in one pivot without involving the shipping costs or the dual variables:

$$(T_1, \vec{x}, \vec{z}) \rightarrow (T_2, \tilde{x}, \tilde{z}).$$

Exercise 14.11



Tree solution:

Flow balance at

- (a): $-X_{ba} = -1 \Rightarrow X_{ba} = 1$
- (b): $X_{ba} + X_{bc} = 1 \Rightarrow X_{bc} = 1 - X_{ba} = 0$
- (c): $-X_{dc} - X_{bc} = -1 \Rightarrow X_{dc} = 1 - X_{bc} = 1 - 0 = 1$

primal feasible

Dual variables y: Let a be the root node ($y_a = 0$),
use $y_j - y_i = c_{ij}$ for $(i,j) \in T_1$.

edges in T_1:

- (b,a): $y_a - y_b = 3 \Rightarrow y_b = -3$
- (b,c): $y_c - y_b = -7 \Rightarrow y_c + 3 = -7 \Rightarrow y_c = -10$
- (d,c): $y_c - y_d = -9 \Rightarrow -10 - y_d = -9 \Rightarrow y_d = -1$

Dual slack variables: $y_j - y_i + z_{ij} = c_{ij}$ for $(i,j) \notin T_1$:

(d,a): $y_a - y_d + z_{da} = c_{da}$

$1 + z_{da} = -2 \Rightarrow z_{da} = -3$ not dual feasible

We make a pivot where X_{da} enters the basis

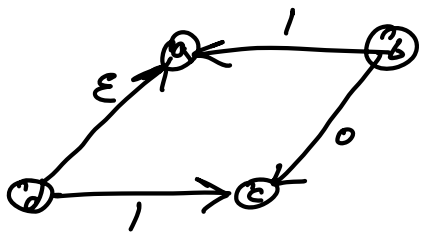
(X_{da} increase to ε)

$\tilde{X}_{da} = \varepsilon$

$\tilde{X}_{ba} = 1 - \varepsilon$

$\tilde{X}_{bc} = \varepsilon$

$\tilde{X}_{dc} = 1 - \varepsilon$



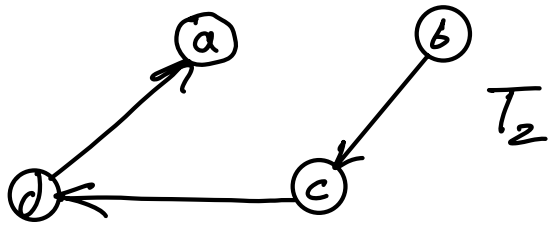
can increase ε to 1, X_{ba} leaves (not unique).

$x_{da} = 1$ (entered)

$x_{ba} = 0$ (left)

$x_{bc} = 1$

$x_{dc} = 0$



Dual variables y_i:

edges in T_2:

(d,a): $y_a - y_d = -2 \Rightarrow y_d = 2$

(b,c): $y_c - y_b = -7 \Rightarrow y_b = y_c + 7 = 0$

(d,c): $y_c - y_d = -9 \Rightarrow y_c = y_d - 9 = -7$

Dual slack variables:

edges outside T_2: (b,a): $y_a - y_b + z_{ba} = c_{ba}$

$0 - 0 + z_{ba} \geq 3 \Rightarrow z_{ba} = 3$

$\Rightarrow$ dual feasible. $\Rightarrow$ X optimal.

Exercise 14.14

```
remaining_nodes = nodes \ {node1}
added_nodes = {node1}
spantree = ∅
connected = true
while remaining_nodes not empty and connected:
    found = false
    for (i,j) ∈ A
        if i in added_nodes and j in remaining_nodes:
            # add (i,j) to spantree
            # add j to added_nodes, remove j from remaining_n
            found = true
        elif j in added_nodes and i in remaining_nodes:
            # add (i,j) to spantree
            # add i to added_nodes, remove j from from remaining_n
            found = tree

    if not found:
        connected = false
```

Exam 2024

1. a)

$$\begin{aligned} \xi &= x_1 + 2x_2 \\ w_1 &= 9 - 3x_1 - x_2 \\ w_2 &= 1 + x_1 - x_2 \end{aligned}$$

ratios: $1/9$, 1 — biggest, w_2 leaves

$x_2 = 1 + x_1 - w_2$

$$\begin{aligned} \xi &= 2 + 3x_1 - 2w_2 \\ w_1 &= 8 - 4x_1 + w_2 \\ x_2 &= 1 + x_1 - w_2 \end{aligned}$$

ratios: $1/2$, -1 — biggest, w_1 leaves

$x_1 = 2 - \frac{1}{4}w_1 + \frac{1}{4}w_2$

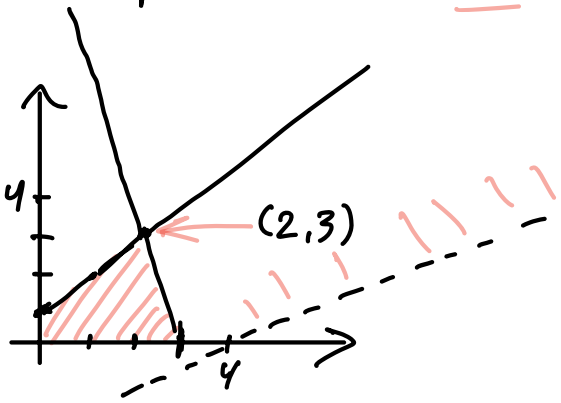
$$\begin{aligned} \xi &= 8 - \tfrac{3}{4}w_1 - \tfrac{5}{4}w_2 \\ x_1 &= 2 - \tfrac{1}{4}w_1 + \tfrac{1}{4}w_2 \\ x_2 &= 3 - \tfrac{1}{4}w_1 - \tfrac{3}{4}w_2 \end{aligned}$$

optimal! value: 8

$\vec{x} = (x_1, x_2) = (2, 3)$

$\vec{w} = (w_1, w_2) = (0, 0)$

b)



c) (i): P': add constraint $x_1 - 2x_2 \leq 4$ $\quad (x_2 \geq \frac{1}{2}x_1 - 2)$

$x_1 = 4 \Rightarrow x_2 = 0$

see that this new constraint is redundant, i.e., it is satisfied for P any way.
same feasible region, and same objective value.

(ii) $x_1 + 3x_2 = 2 - \frac{1}{4}w_1 + \frac{1}{4}w_2 = 11 - w_1 - 2w_2$

$+ 9 - \frac{3}{4}w_1 - \frac{9}{4}w_2$

so dictionary still optimal with new objective.

$(x_1, x_2) = (2, 3)$, $\eta = 11$

d) dual problem of (P)

$$\begin{aligned} \min \quad & 9y_1 + y_2 \\ \text{s.t} \quad & 3y_1 - y_2 \geq 1 \\ & y_1 + y_2 \geq 2 \\ & y_i \geq 0 \end{aligned}$$

e) optimal dual dictionary:

$$\begin{aligned} -\eta &= -8 - 2z_1 - 3z_2 \\ y_1 &= \tfrac{3}{4} + \tfrac{1}{4} z_1 + \tfrac{1}{4} z_2 \\ y_2 &= \tfrac{5}{4} - \tfrac{1}{4} z_1 + \tfrac{3}{4} z_2 \end{aligned}$$

$\eta = 8$, $\vec{y} = (y_1, y_2) = (\frac{3}{4}, \frac{5}{4})$, $\vec{z} = (z_1, z_2) = (0,0)$.

2)

$$\begin{aligned} \min \quad & 3x_1 - 4x_2 - 2x_3 \\ \text{s.t.} \quad & -x_1 + 2x_2 + 3x_3 \geq 2 \\ & 2x_1 - x_2 = 5 \\ & x_1, x_3 \geq 0 \end{aligned}$$

→

$$\begin{aligned} \max \quad & -3x_1 + 4x_2 + 2x_3 \\ & x_1 - 2x_2 - 3x_3 \leq -2 \\ & 2x_1 - x_2 \leq 5 \\ & -2x_1 + x_2 \leq -5 \\ & x_1, x_3 \geq 0 \end{aligned}$$

$x_2 = y - z$, $y, z \geq 0$

$$\begin{aligned} \max \quad & -3x_1 + 4y - 4z + 2x_3 \\ & x_1 - 2y + 2z - 3x_3 \leq -2 \\ & 2x_1 - y + z \leq 5 \\ & -2x_1 + y - z \leq -5 \\ & x_1, x_2, x_3, y, z \geq 0 \end{aligned}$$

infeasible dictionary

⇓

2-phase method.

3 a) a polytope P is convex: $P = \text{conv}(\{x_i\}_{i=1}^n)$

$x, y \in P$: $x = \sum_{i=1}^n \mu_i x_i$, $y = \sum_{i=1}^n \nu_i x_i$

$$(1-\lambda)x + \lambda y = (1-\lambda)\sum_{i=1}^n \mu_i x_i + \lambda \sum_{i=1}^n \nu_i x_i$$

$$= \sum_{i=1}^n \underbrace{((1-\lambda)\mu_i + \lambda \nu_i)}_{\text{sum to } \sum_{i=1}^n((1-\lambda)\mu_i + \lambda\nu_i)} x_i$$

$$= (1-\lambda)\sum_{i=1}^n \mu_i + \lambda \sum_{i=1}^n \nu_i$$

$$= (1-\lambda)\cdot 1 + \lambda \cdot 1 = 1-\lambda+\lambda = 1,$$

It follows that $(1-\lambda)x + \lambda y \in P \Rightarrow P$ convex.

b) $S = \{x \in \mathbb{R}^n : A_1 x \le b_1, A_2 x \le b_2\}$ convex ?

Let $x, y \in S$:

$$A_i((1-\lambda)x + \lambda y) = (1-\lambda) A_i x + \lambda A_i y$$

$$\le (1-\lambda) b_i + \lambda b_i = b_i$$

This shows that $(1-\lambda)x + \lambda y \in S \Rightarrow S$ convex

4

a) Tree solution:

flow balance at

(a) (leaf) : $x_{ac} = 3$

(d) (leaf) : $x_{db} = 4$

(c) : $x_{cb} - x_{ac} = -2 \Rightarrow x_{cb} = 1$

(all other $x_{ij} = 0$)

b) dual variables: $y_j - y_i = c_{ij}$ for $(i,j) \in T$, $y_a = 0$ (root node)

(a,c) : $y_c - y_a = c_{ac} \Rightarrow y_c = 2$

(c,b) : $y_b - y_c = c_{cb} \Rightarrow y_b - 2 = 4 \Rightarrow y_b = 6$

(d,b) : $y_b - y_d = c_{db} \Rightarrow 6 - y_d = 1 \Rightarrow y_d = 5$

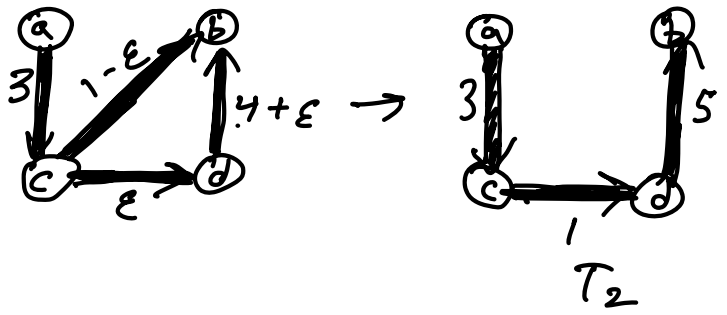
dual slack variables $y_j - y_i + z_{ij} = c_{ij}$ for $(i,j) \notin T_1$:

(a,d): $y_d - y_a + z_{ad} = c_{ad} \Rightarrow 5 - 0 + z_{ad} = 6 \Rightarrow z_{ad} = 1$

(b,a): $y_a - y_b + z_{ba} = c_{ba} \Rightarrow 0 - 6 + z_{ba} = 3 \Rightarrow z_{ba} = 9$

(c,d): $y_d - y_c + z_{cd} = c_{cd} \Rightarrow 5 - 2 + z_{cd} = 2 \Rightarrow z_{cd} = -1$

not optimal, x_{cd} enters
increase x_{cd} with ε
flow in cycle changes.



T_2

$x_{ac} = 3, x_{cd} = 1$

$x_{db} = 5$

dual variables:

(a,c): $y_c - y_a = c_{ac} \Rightarrow y_c = 2$

(c,d): .... $\Rightarrow y_d = 4$

(d,b): ... $\Rightarrow y_b = 5$

dual slack variables:

(b,a): $y_a - y_b + z_{ba} = c_{ba} \Rightarrow z_{ba} = 8$

(c,b): .... $z_{cb} = 1$

(a,d): .... $z_{ad} = 2$

optimal!

value: $c_{ac} x_{ac} + c_{cd} x_{cd} + c_{db} x_{db}$

$= 2 \cdot 3 + 2 \cdot 1 + 1 \cdot 5 = 6 + 2 + 5 = 13$