Oblig 1 MAT3100

Deadline 13. march 2025, 1430

March 21, 2025

Consider the problem

$$\begin{array}{rrrrrl} \max & x_1 & + & 3x_2 & & \\ \text{s.t.} & -x_1 & + & x_2 & \leq & 2 \\ & x_1 & - & x_2 & \leq & 2 \\ & x_1 & & & \leq & 4 \\ & & & x_2 & \leq & 4 \\ & & & x_1, x_2 & \geq & 0 \end{array} \tag{1}$$

a)

Write down a 4×2-matrix A and vectors $\mathbf{b}$ and $\mathbf{c}$ so that (2) can be written on the form

$$\begin{aligned} \max \ & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} \ & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Solution: We have that

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ 4 \\ 4 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

b)

Solve the problem using the Simplex method and the largest coefficient rule.
Solution: The initial dictionary is

$$\begin{array}{rcrcrcr} \eta & = & 0 & + & x_1 & + & 3x_2 \\ \hline w_1 & = & 2 & + & x_1 & - & x_2 \\ w_2 & = & 2 & - & x_1 & + & x_2 \\ w_3 & = & 4 & - & x_1 & & \\ w_4 & = & 4 & & & - & x_2 \end{array}$$

This is feasible. If we apply the largest coefficient rule, x_2 will be the entering variable. The ratios are $1/2$, $-1/2$, 0 and $1/4$. The biggest is $1/2$, so that w_1 is

the leaving variable. Rewriting the second constraint as $x_2 = 2 + x_1 - w_1$, and inserting this in the dictionary, we obtain

$$\begin{array}{rcrcrcr} \eta & = & 6 & + & 4x_1 & - & 3w_1 \\ \hline x_2 & = & 2 & + & x_1 & - & w_1 \\ w_2 & = & 4 & & & - & w_1 \\ w_3 & = & 4 & - & x_1 & & \\ w_4 & = & 2 & - & x_1 & + & w_1 \end{array}$$

Now x_1 is entering, and the ratios are $-1/2$, 0, $1/4$, and $1/2$. $1/2$ is biggest so that w_4 leaves. We rewrite the fourth constraint as $x_1 = 2 - w_4 + w_1$, and insert in the dictionary:

$$\begin{array}{rcrcrcr} \eta & = & 14 & - & 4w_4 & + & w_1 \\ \hline x_2 & = & 4 & - & w_4 & & \\ w_2 & = & 4 & & & - & w_1 \\ w_3 & = & 2 & + & w_4 & - & w_1 \\ x_1 & = & 2 & - & w_4 & + & w_1 \end{array}$$

Now w_1 is entering, and the ratios are 0, $1/4$, $1/2$, and $-1/2$. w_3 thus leaves, and by inserting $w_1 = 2 + w_4 - w_3$ we get

$$\begin{array}{rcrcrcr} \eta & = & 16 & - & 3w_4 & - & w_3 \\ \hline x_2 & = & 4 & - & w_4 & & \\ w_2 & = & 2 & - & w_4 & + & w_3 \\ w_1 & = & 2 & + & w_4 & - & w_3 \\ x_1 & = & 4 & & & - & w_3 \end{array}$$

This dictionary is optimal. The optimal value is 16, and $\mathbf{x} = (4, 4)$ is a unique optimizer.

c)

Sketch the feasible region of the problem (2). In your drawing, indicate the (directed) path taken by the basic feasible solutions in the iterations of simplex from **b)**. If you instead applied Blands rule, what would this path be?
Solution: The feasible region can be drawn by connecting the points $(0, 0)$, $(1, 0)$, $(2, 1)$, $(2, 2)$, $(1, 2)$, $(0, 1)$ and $(0, 0)$ sequentially by line segments, and highlighting its interior. The points $(0, 0)$, $(5, 0)$, $(4, 2)$, $(2, 4)$, $(0, 5)$ are the extreme points of the feasible region. They can also be found as the basic feasible solutions to the problem by going through all possible choices of bases (that was not required here though), of which some are feasible and others not. At the start of simplex, the basic solution is $(0, 0)$. After the first iteration the basic solution is $(0, 1)$, then at $(1, 2)$, and finally at $(2, 2)$.

Blands rule would make a different choice for leaving variable in the first iteration: x_1 would enter, and the first iteration would end at $(1, 0)$. In the next iteration it is only possible to go to $(2, 1)$ (as simplex follows the border of the feasible region, and since the objective can't decrease), and then to $(2, 2)$.

d)

We consider the same constraints as in **a)**, but change the objective function to x_2. Find an optimal solution to this modified problem. Is the optimal solution

unique? If not, find an expression for the general optimal solution. You can use your drawing of the feasible region from **c)** as a guide.
Solution: The starting dictionary is the same as in **b)**, with the objective replaced with $\eta = x_2$. The first iteration gives the same entering variable as the largest coefficient rule, so that the first iteration ends up with the same nonbasic variables, with the new objective $x_2 = 2 + x_1 - w_1$. Now x_1 would enter, just as in the second iteration of **b)**. The objective changes to $x_2 = 4 - w_4$, so that in this case we obtain an optimal dictionary already after 2 iterations. In this optimal dictionary w_1 can vary however, and we see that the basic variables remain nonnegative for $0 \leq w_1 \leq 2$. This means that the general optimal solution is $\mathbf{x} = (2 + w_1, 4)$, i.e., the line segment from $(2, 4)$ to $(4, 4)$. This is also easily inferred from the drawing of the feasible region.

e)

Write down the dual problem of (2), as well as the initial dual dictionary. Is this dictionary feasible?
Solution: The dual problem is to minimize $\mathbf{b}^T\mathbf{y}$ subject to the constraints $A^T\mathbf{y} \geq \mathbf{c}$, and $\mathbf{y} \geq \mathbf{0}$, with A, $\mathbf{b}$, and $\mathbf{c}$ as in **a)**. We thus get

$$\begin{array}{lrcrcrcrcl} \min & 2y_1 & + & 2y_2 & + & 4y_3 & + & 4y_4 & & \\ \text{s.t.} & -y_1 & + & y_2 & + & y_3 & & & \geq & 1 \\ & y_1 & - & y_2 & & & + & y_4 & \geq & 3 \\ & & & & & & & y_1, y_2, y_3, y_4 & \geq & 0 \end{array} \tag{2}$$

The dual dictionary is (take the negative transpose)

$$\begin{array}{rcrcrcrcrcr} -\xi & = & & - & 2y_1 & - & 2y_2 & - & 4y_3 & - & 4y_4 \\ \hline z_1 & = & -1 & - & y_1 & + & y_2 & + & y_3 & & \\ z_2 & = & -3 & + & y_1 & - & y_2 & & & + & y_4 \end{array}$$

The initial basic solution is $(0, 0, 0, 0)$. We see that the dual dictionary is infeasible (this is equivalent to that the primal dictionary is not optimal).

f)

Write down the dual dictionaries corresponding to the primal dictionaries you obtained when applying the simplex method in **b)**. What are the basic solutions obtained by these dual dictionaries?
Solution: The dual dictionaries corresponding to the dictionaries in **b)**:

$$\begin{array}{rcrcrcrcrcr} -\xi & = & -6 & - & 2z_2 & - & 4y_2 & - & 4y_3 & - & 2y_4 \\ \hline z_1 & = & -4 & - & z_2 & & & + & y_3 & + & y_4 \\ y_1 & = & 3 & + & z_2 & + & y_2 & & & - & y_4 \end{array}$$

After the first iteration y_2, y_3, y_4 are nonbasic, while y_1 is basic, and its value is 3 in the feasible solution. This gives the point $(3, 0, 0, 0)$.

$$\begin{array}{rcrcrcrcrcr} -\xi & = & -14 & - & 4z_2 & - & 4y_2 & - & 2y_3 & - & 2z_1 \\ \hline y_4 & = & 4 & + & z_2 & & & - & y_3 & + & z_1 \\ y_1 & = & -1 & & & + & y_2 & + & y_3 & - & z_1 \end{array}$$

After the second iteration y_2, y_3 are nonbasic while y_1, y_4 are basic, with values $y_1 = -1$ and $y_4 = 4$. This gives the point $(-1, 0, 0, 4)$.

$$\begin{array}{rcrcrcrcrcr} -\xi & = & -16 & - & 4z_2 & - & 2y_2 & - & 2y_1 & - & 4z_1 \\ \hline y_4 & = & 3 & + & z_2 & + & y_2 & - & y_1 & & \\ y_3 & = & 1 & & & - & y_2 & + & y_1 & + & z_1 \end{array}$$

After the third iteration y_1, y_2 are nonbasic while y_3, y_4 are basic, with values $y_3 = 1$ and $y_4 = 3$. This gives the point $\mathbf{y} = (0, 0, 1, 3)$, which is optimal

g)

Find the slack variables for the primal and dual optimal solutions. State what is meant by complementary slackness, and verify that this property holds for the optimal solutions you have found.
Solution: At the optimum for the dual problem, z_1, z_2 are nonbasic, so that $\mathbf{z} = (0, 0)$. It follows that

$$\mathbf{x} \cdot \mathbf{z} = (4, 4) \cdot (0, 0) = (0, 0).$$

At the optimum for the primal problem, $\mathbf{w} = (2, 2, 0, 0)$. It follows that

$$\mathbf{y} \cdot \mathbf{w} = (0, 0, 1, 3) \cdot (2, 2, 0, 0) = (0, 0),$$

and we have verified complementary slackness.

The following code verifies the computations above

```
% b)
D = [0  1 3; 2 1 -1; 2  -1 1; 4  -1  0; 4  0 -1]
D = pivot(D, 2, 1)
D = pivot(D, 1, 4)
D = pivot(D, 2, 3)

% d)
D = [0  0 1; 2 1 -1; 2  -1 1; 4  -1  0; 4  0 -1]
D = pivot(D, 2, 1)
D = pivot(D, 1, 4)

% f)
D = [0 -2 -2 -4 -4; -1 -1 1 1 0; -3 1 -1 0 1]
D = pivot(D, 1, 2)
D = pivot(D, 4, 1)
D = pivot(D, 3, 2)
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