Oblig 2 MAT3100

Deadline 24. march 2025, 1430

May 9, 2025

This compulsory exercise consists of two problems, with 3 subproblems each. Note that you will pass as long as you hand in the exercise on canvas. Feedback will be given on all solutions you submit.

Problem 1 (Game theory)

For this first problem you may need to consult the lecture notes on game theory from 25. march (see the schedule of the course), as the material on pure minmax- and maxmin strategies can not be found in the book. We consider the matrix game with payoff matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -2 & -2 \\ 1 & -2 & 0 \end{pmatrix}$$

a)

Find the best pure strategies for the row- and column player (i.e., the minmax and maxmin strategies), and their corresponding payoffs (V^* and V_*). Show that the game does not have a value.

Solution: If we take the maxima over all the rows, we get 1, 2, and 1. The minimum of these values is $V^* = 1$. This gives that the pure minmax strategy for the row player is $i = 1$ or $i = 3$, with a payoff of 1.

If we take the minima over the columns we get 0, -2, and -2. The maximum of these values is $V_* = 0$. This gives that the best pure strategy for the column player (the pure maxmin strategy) is $j = 1$, with a payoff of 0. Since $V^* \neq V_*$, the game does not have a value, i.e., no pure strategies are optimal.

b)

State the Minimax Theorem for matrix games. Explain why the pure strategies found above are not optimal, so that both players should consider using a randomised strategy instead.

Solution: The minimax theorem states that there exist strategies x^* and y^* so that

$$\max_x (y^*)^T A x = \min_y y^T A x^*,$$

and that strategies satisfying this are mutually optimal. If the pure maxmin/minmax strategies were optimal they would satisfy this, i.e. we would have $V^* = V_*$.

Since this is not the case, they cannot be optimal. The optimal strategies must thus be randomised strategies.

c)

Let $\mathbf{x}^* = (3/4, 1/4, 0)$ and $\mathbf{y}^* = (3/4, 0, 1/4)$ be randomised strategies for the column player and row player, respectively. Prove that these are optimal strategies, and find the value of the game. Is the game fair? If not, which player wins in the long run?
Solution: We have that

$$A\mathbf{x}^* = \begin{pmatrix} 0 & 1 & -2 \\ 2 & -2 & -2 \\ 1 & -2 & 0 \end{pmatrix} \begin{pmatrix} 3/4 \\ 1/4 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1 \\ 1/4 \end{pmatrix},$$

and this gives

$$\min_{\mathbf{y}} \mathbf{y}^T A\mathbf{x}^* = \min_{\mathbf{y}} \mathbf{y}^T \begin{pmatrix} 1/4 \\ 1 \\ 1/4 \end{pmatrix} = 1/4,$$

where the minimum is taken over all stochastic vectors. We also get

$$(\mathbf{y}^*)^T A = \begin{pmatrix} 3/4 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 1 & -2 \\ 2 & -2 & -2 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & -3/2 \end{pmatrix},$$

and this gives

$$\max_{\mathbf{x}} (\mathbf{y}^*)^T A\mathbf{x} = \begin{pmatrix} 1/4 & 1/4 & -3/2 \end{pmatrix} \mathbf{x} = 1/4.$$

Since these are equal, the two strategies are optimal, and the value of the game is $1/4$. Since this is nonzero, the game is not fair. Since the value of the game is positive, the column player wins in the long run.

Problem 2 (Convexity)

a)

Explain what a polyhedron is, and what a polytope is. Show also that the intersection of two polyhedra is again a polyhedron, and that the intersection of two polytopes is again a polytope.
Solution: A polytope is a set on the form $\{x : A\mathbf{x} \leq \mathbf{b}\}$. Let

$$P = \{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\} \qquad Q = \{\mathbf{x} : C\mathbf{x} \leq \mathbf{d}\}.$$

Then

$$P \cap Q = \left\{ \mathbf{x} : \begin{pmatrix} C \\ D \end{pmatrix} \mathbf{x} \leq \begin{pmatrix} \mathbf{b} \\ \mathbf{d} \end{pmatrix} \right\},$$

which is a polyhedron. For the second part, we know that a polytope is a bounded polyhedron (corollary 6 in a mini-introduction to convexity). The intersection of two bounded sets is again bounded. Since also the intersection of

two polyhedra has been shown to be a polyhedron, it follows that the intersection of two bounded polyhedra is again a bounded polyhedron, so that the intersection of two polytopes is again a polytope.

In the following, $A + B$ is the set consisting of all points on the form $a + b$ with a in the set A, b in the set B.

b)

Show that $A + B$ is a convex set when A and B are convex sets.
Solution: Let $a_1 + b_1, a_2 + b_2 \in A + B$, where $a_1, a_2 \in A$, $b_1, b_2 \in B$. Then

$$(1-\lambda)(a_1+b_1) + \lambda(a_2+b_2) = ((1-\lambda)a_1 + \lambda a_2) + ((1-\lambda)b_1 + \lambda b_2) \in A + B$$

since $(1-\lambda)a_1 + \lambda a_2 \in A$, $(1-\lambda)b_1 + \lambda b_2 \in B$ because of convexity of A and B. It follows that $A + B$ is convex.

c) (difficult)

Explain what the convex hull of a set is. Also, prove that

$$\text{conv}(A + B) = \text{conv}(A) + \text{conv}(B)$$

when A and B are subsets of $\mathbb{R}^n$ (conv denotes convex hull).
Hint: Consider the sum $\sum_{j,k} \lambda_j \mu_k (a_j + b_k)$ where $a_j \in A$, $b_k \in B$ and $\lambda_j \geq 0$, $\mu_k \geq 0$ and $\sum_j \lambda_j = 1$ and $\sum_k \mu_k = 1$.
Solution: We have that

$$\begin{aligned}
\sum_{j,k} \lambda_j \mu_k (a_j + b_k) &= \sum_{j,k} \lambda_j a_j \mu_k + \sum_{j,k} \lambda_j \mu_k b_k \\
&= \left(\sum_k \mu_k\right)\left(\sum_j \lambda_j a_j\right) + \left(\sum_j \lambda_j\right)\left(\sum_k \mu_k b_k\right) \\
&= \sum_j \lambda_j a_j + \sum_k \mu_k b_k.
\end{aligned}$$

The right hand side is a general element in $\text{conv}(A) + \text{conv}(B)$. Since the left hand side is a convex combination of elements in $A + B$, it follows that $\text{conv}(A) + \text{conv}(B) \subseteq \text{conv}(A + B)$. The other way, $\text{conv}(A) + \text{conv}(B)$ is clearly convex (this follows from **b)**), so that it equals its convex hull. Therefore

$$\text{conv}(A + B) \subseteq \text{conv}(\text{conv}(A) + \text{conv}(B)) = \text{conv}(A) + \text{conv}(B),$$

and the result follows.