UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT 3100 — Linear optimization

Day of examination: June 4, 2026

Examination hours: 15.00 – 19.00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

Problem 1

Consider the LP problem:

$$\begin{array}{lrcrcrcr} \max & x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & + & x_3 & \leq & 10 \\ & 2x_1 & + & x_2 & & & \leq & 12 \\ & & & x_2 & + & 3x_3 & \leq & 9 \\ & & & & x_1, x_2, x_3 \geq 0. & & & \end{array} \tag{1}$$

1a

Solve problem (1) using the simplex algorithm.

Solution: Simplex: initial dictionary:

$$\begin{array}{rcrcrcrcr} \zeta & = & 0 & + & x_1 & + & 4x_2 & & 3x_3 \\ \hline w_1 & = & 10 & - & x_1 & - & x_2 & - & x_3 \\ w_2 & = & 12 & - & 2x_1 & - & x_2 & & \\ w_2 & = & 9 & & & - & x_2 & + & 3x_3 \end{array}$$

1. pivot: x_2 enters basis, and w_2 leaves. This gives:

$$\begin{array}{rcrcrcrcr} \zeta & = & 36 & + & x_1 & - & 4w_3 & - & 9x_3 \\ \hline w_1 & = & 1 & - & x_1 & + & w_3 & + & 2x_3 \\ w_2 & = & 3 & - & 2x_1 & + & w_3 & + & 3x_3 \\ x_2 & = & 9 & & & - & w_3 & - & 3x_3 \end{array}$$

(Continued on page 2.)

2. pivot: x_1 enters basis, and w_1 leaves. This gives:

$$\begin{array}{rcrcrcrcr} \zeta & = & 37 & - & w_1 & - & 3w_3 & - & 7x_3 \\ \hline x_1 & = & 1 & - & w_1 & + & w_3 & + & 2x_3 \\ w_2 & = & 1 & + & 2w_1 & - & w_3 & - & x_3 \\ x_2 & = & 9 & & & - & w_3 & - & 3x_3 \end{array}$$

So, an optimal solution is: $x_1 = 1$, $x_2 = 9$, $x_3 = 0$, $w_1 = 0$, $w_2 = 1$, $w_3 = 0$; the optimal value is 37. ◇

Consider the LP problem obtained from problem (1) dropping the variable x_3, so this problem is

$$\begin{array}{lrcrcl} \max & x_1 & + & 4x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & \leq & 10 \\ & 2x_1 & + & x_2 & \leq & 12 \\ & & & x_2 & \leq & 9 \\ & & & x_1, x_2 \geq 0. & & \end{array} \tag{2}$$

1b

Illustrate the feasible region of (2) in the plane, and explain geometrically why $x_1 = 1$, $x_2 = 9$ is an optimal solution.

Solution. Note: $x_3 = 0$ in the unique optimal solution of problem (1), so these two problems are equivalent. Geometry: Draw the 5 lines corresponding to the 2 coordinate axes, and the 3 equations $x_1 + x_2 = 10$, $2x_1 + x_2 = 12$, and $x_2 = 9$. The feasible region is the polytope with extreme points (vertices) $(0, 0)$, $(6, 0)$, $(2, 8)$, $(1, 9)$ and $(0, 9)$. Then we see that $(1, 9)$ is optimal by drawing level set lines $x_1 + 4x_2 = \alpha$ (start with α large, and make paralells until the feasible region is hit). Also, one could just compute the objective functions in these 5 extreme points and compare values. ◇

Consider the LP problem given by the following dictionary

$$\begin{array}{rcrcrcrcr} \zeta & = & 3 & & & - & 2x_2 & - & x_3 \\ \hline x_4 & = & 4 & - & x_1 & + & 2x_2 & - & x_3 \\ x_5 & = & 1 & & & - & x_2 & + & x_3 \end{array} \tag{3}$$

1c

Find all optimal solutions to this problem. (Hint: there is no need for the simplex algorithm.)

Solution. (Here it is implicit that alle variables are ≥ 0; as usual. This was also informed early at the exam.) We see that $x_4 = 4$, $x_5 = 1$, $x_1 = x_2 = x_3 = 0$ is an optimal solution. Since the reduced cost of x_1 is 0, and the other two are negative, we get all other optimal solution by increasing x_1 from 0 up to 4. Thus the optimal solutions are $x_1 = t$, $x_4 = 4 - t$, $x_5 = 1$, $x_2 = x_3 = 0$ for each $t \in [0, 4]$. ◇

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Problem 2

2a

Formulate the duality theorem of linear optimization (LP) when the primal problem is

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & \\ & Ax \le b \\ & x \ge 0. \end{array}$$

Solution: The dual problem is

$$\begin{array}{ll} \min & b^T y \\ \text{subject to} & \\ & A^T y \ge c \\ & y \ge 0. \end{array}$$

Duality theorem: If at least one of the two problems has a feasible solution, then the optimal value is the same in the two problems. (Here the optimal value can be ∞ or $-\infty$, corresponding to one problem being infeasible and the other unbounded). ◇

2b

Find the dual problem of the LP problem (2). Show that $y_1 = 1$, $y_2 = 0$, $y_3 = 3$ is an optimal solution of this dual.

Solution: The dual is

$$\begin{array}{lrcrcrcl} \min & 10y_1 & + & 12y_2 & + & 9y_3 & & \\ \text{subject to} & & & & & & & \\ & y_1 & + & 2y_2 & & & \ge & 1 \\ & y_1 & + & y_2 & + & y_3 & \ge & 4 \\ & & & & y_1, y_2, y_3 \ge 0. & & & \end{array}$$

$y = (y_1, y_2, y_3) = (1, 0, 3)$ is feasible, i.e., satisfies all the constraints (nonnegative and the first two inequalities hold with equality). Also, the corresponding value of the objective function is $10 \cdot 1 + 9 \cdot 3 = 37$ which is the optimal value in (2). By weak duality y must then be an optimal solution of the dual. ◇

Problem 3

Let D be the directed graph indicated in Figure 1 with vertices (nodes) v_1, v_2, v_3 and v_4. Consider the minimum cost network flow problem in D where the supply/demand is given by $b(v_1) = 2$ (so in vertex v_1 flow out minus flow

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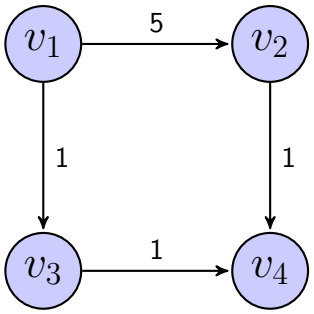


Figure 1: The directed graph D.

in is 2), $b(v_2) = 1$, $b(v_3) = 0$, $b(v_4) = -3$. (We write $b(v)$ for b_v so it is easier to read.) The numbers along the edges are the costs. There are no upper capacities on flows.

3a

Let T be the tree with edges (v_1, v_2), (v_2, v_4) and (v_3, v_4). Compute the tree solution x that corresponds to the tree T. Also compute its cost.

Solution: Must have $x(v_1, v_3) = 0$ as this edge is not in T. Then from flow balance equations:

$$x(v_1, v_2) = 2,\ x(v_2, v_4) = 3,\ x(v_3, v_4) = 0 \text{ and } \ x(v_1, v_3) = 0.$$

The corresponding cost is $5 \cdot 2 + 3 \cdot 1 = 13$. ◇

3b

Find an optimal solution of this network flow problem, and its optimal value.

Solution: There exists an optimal tree solution, and there are only 4 such (as there are 4 spanning trees). We must have positive flow in (v_2, v_4) so that edge is in the optimal tree. By checking we see optimal tree solution is for T^* given by (v_1, v_3), (v_3, v_4), (v_2, v_4), with x^* given by

$$x^*(v_1, v_2) = 0,\ x^*(v_1, v_3) = 2,\ x^*(v_3, v_4) = 2 \text{ and } x^*(v_2, v_4) = 1.$$

The optimal cost is $2+2+1 = 5$. Alternative: make one pivot in the network simplex algorithm. ◇

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Problem 4

Let $P \subseteq \mathbb{R}^5$ be the set of all solutions to the linear system

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & & & & & & \leq 1 \\ & & x_2 & + & x_3 & & & & \leq 1 \\ & & & & x_3 & + & x_4 & & \leq 1 \\ & & & & & & x_4 & + & x_5 \leq 1 \\ x_1 & & & & & & & + & x_5 \leq 1 \end{array}$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0.$$

4a

Give the definition of an extreme point of a polyhedron. [This gives 3 points] Show that $x^* = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is an extreme point of P. [This gives 7 points]

Solution: Definition: A point x in a polyhedron Q is an extreme point if $x = (1/2)y + (1/2)z$ for $y, z \in Q$ implies that $x = y = z$.

Concerning x^*: First we see that x^* satisfies all the inequalities defining P so $x^* \in P$. Observe that x^* satisfies all the five $\leq$-inequalities with equality. Assume $x = (1/2)y + (1/2)z = (1/2)(y+z)$ where $y, z \in P$. Then $y_1 + y_2 \leq 1$ and $z_1 + z_2 \leq 1$, so if one of these inequalities were strict, we would get

$$x_1 + x_2 = (1/2)(y_1 + z_1 + y_2 + z_2) < (1/2) \cdot 2 = 1,$$

a contradiction. Thus, $y_1 + y_2 = 1$ and $z_1 + z_2 = 1$. Similarly, we can show that each of the five $\leq$-inequalities for P hold with equality for both y and z. Thus each of x, y and z satisfy the linear system $Hw = e$ where $e \in \mathbb{R}^5$ is the all ones vector, and

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

But H is invertible (non-singular), so $Hw = e$ has a unique solution. Therefore $x = y = z$, and this proves that x is an extreme point. $\diamond$

4b

Let S be the set of *all* the extreme points of P (you don't need to find them), and let Q be the convex hull of S. What is the relation between P and Q? Give reference to a theorem/corollary from convexity on this.

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Solution: We have $P = Q$. This is a consequence of convexity: see the Main theorem for polyhedra, and the comments right after. A bounded polyhedron is a polytope and it is the convex hull of its extreme points. $\diamond$

Problem 5

Consider the LP where the right-hand sides are identically zero

$$\begin{array}{lll} \max & \sum_{j=1}^n c_j x_j & \\ \text{subject to} & & \\ & \sum_{j=1}^n a_{ij} x_j \;\leq\; 0 & (i = 1, 2, \ldots, m), \\ & x_j \;\geq\; 0 & (j = 1, 2, \ldots, n). \end{array}$$

Show that either $x = 0$ (i.e., $x_j = 0$ for all j) is an optimal solution or else the problem is unbounded.

Solution: As $x = 0$ is a feasible soution, the LP either has an optimal solution or it is unbounded. There are two possibilities.

Case 1: Every feasible x (i.e., x satisfying the constraints in the LP) satisfies $\sum_j c_j x_j \leq 0$. Then clearly $x = 0$ is an optimal solution, with optimal value 0.

Case 2: Otherwise, there is a feasible x such that $a := \sum_j c_j x_j > 0$. Let $z = \lambda x$ for some $\lambda > 0$, then $z = (z_1, z_2, \ldots, z_n)$ is also a feasible solution and

$$\sum_j c_j z_j = \lambda \sum_j c_j x_j = \lambda a \to \infty \quad \text{when} \quad \lambda \to \infty.$$

Thus, the problem is unbounded.

Alternatively: can be shown by the simplex algorithm, or by the main theorem for polyhedra, or by discussing the dual problem (wheter it is feasible or not). $\diamond$

Good luck!