Linear programming (LP): getting started

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Basic stuff:

- LP (linear programming, linear optimization) is the core of optimization.
- LP: *maximize or minimize a linear function subject to linear constraints* (linear equations and linear inequalities).
- example: max $2x_1 - x_2 + x_3$
  subject to: $x_1 + 5x_2 \leq 1$, $x_1 - x_3 = 0$, $x_1 \geq 0$, $0 \leq x_2 \leq 3$.
- objective function: the function we want to maximize, here $2x_1 - x_2 + x_3$.
- LP matrix form: $(*)$ max $c^T x$ s.t. $Ax \leq b$, $x \geq 0$, where A is $m \times n$ matrix etc.
- George B. Dantzig (1914-2005): the father of LP.
- techniques for rewriting any LP to the form $(*)$.
- applications: many practical planning problems; also used to approximate more complicated (nonlinear) problems; linear approximation: min $\|Ax-b\|_\infty$ $\rightarrow$ LP
- simplex algorithm: main method, efficient in practice. Also interior point methods.
- main simplex alg. idea: rewrite LP in terms of linear equations in nonnegative variables, and manipulate the equations by solving for certain variables. Check if the solution can be improved. Repeat.

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- specialized version solves network flow problems efficiently.

Equations and the simplex algorithm:

- Inequalities replaced by equations: $x_1 + 5x_2 \leq 1$ replaced by $x_1 + 5x_2 + w_1 = 1$, $w_1 \geq 0$. So: linear equation: nice. One extra variable, w_1 which must be nonnegative. We do this for each inequality, and get one new variable for each.

- Why? Because we know how to handle linear equations. We get underdetermined equations: more variables than equations, so we *may* have many solutions.

- In general: *in a linear system of equations we use elimination or row operations to solve for some chosen variables.*

- Example:

$$\begin{array}{rcrcrcrcl} x_1 & - & & & x_2 & + & x_3 & & = & 1 \\ -x_1 & + & (1/2)x_2 & & & & & + & x_4 & = & 2 \end{array} \tag{1}$$

solving for x_3, x_4 gives

$$\begin{array}{rcrcrcrcl} x_3 & = & 1 & - & x_1 & + & & x_2 \\ x_4 & = & 2 & + & x_1 & - & (1/2)x_2 \end{array} \tag{2}$$

Nice here: easy to find a solution: choose variables on the right hand side (RHS) to be 0, and solve. So $x_3 = 1$ and $x_4 = 2$, i.e., $x = (x_1, x_2, x_3, x_4) = (0, 0, 1, 2)$.

But an alternative is to solve for, say, x_2 and x_3 in (1): multiply the second eq. by 2 and then add the resulting eq. to the first eq., this gives

$$\begin{array}{rcrcrcrcl} -x_1 & & & & x_3 & + & 2x_4 & = & 5 \\ -2x_1 & + & x_2 & & & + & 2x_4 & = & 4 \end{array} \tag{3}$$

or

$$\begin{array}{rcrcrcrcl} x_3 & = & 5 & + & x_1 & - & 2x_4 \\ x_2 & = & 4 & + & 2x_1 & - & 2x_4 \end{array} \tag{4}$$

(we could change the order of these equations, if we want). So, now we again let variables on the RHS to be zero, which gives $x' = (x'_1, x'_2, x'_3, x'_4) = (0, 4, 5, 0)$. So, both x and x' are solutions of (1), obtained by solving for different sets of variables, using row operations. Note: *all these four sets of linear equations are equivalent.* So: *they have exactly the same set of solutions.*

- the term *dictionary* refers to the form like (2) and (4): the variables we solved for are on the left hand side (LHS), and are called *basic variables*. Those on the RHS are called the *nonbasic variables*.

- *Basic variables*: dependent variables. *Nonbasic variables*: independent variables (we can vary these freely and compute corresponding values for the basic variables).

- Note that *any* choice of values for the nonbasic variables is ok, in the sense that we find corresponding values for the basic variables such that all equations hold. But Dantzig's idea was to keep things simple: *let all the nonbasic variables be zero. Then we read off the values of the basic variables.*

How to handle the objective function:

- Consider the LP:

$$\max f(x) = 3x_1 - x_2 \text{ s.t. } x_1 - x_2 \leq 1,\ -x_1 + (1/2)x_2 \leq 2,\ x_1, x_2 \geq 0.$$

  Adding slack variables x_3, x_4 we obtain (1) with $x_3, x_4 \geq 0$. Say we use basic variables x_2 and x_3, so the rewritten system is (4). Now we want to express f in terms of the nonbasic variables (x_1 and x_4), in order to see if the solution can be improved. So

$$f(x) = 3x_1 - x_2 = 3x_1 - (4 + 2x_1 - 2x_4) = -4 + x_1 + 2x_4.$$

  Currently, we have the solution $x = (0, 4, 5, 0)$ with $f(x) = -4$. We can find a better solution by increasing x_1 and/or x_4. Staying with the *keep things simple* idea, we increase just one of these variables.

- So, now it is a job for you:

  (i) increase x_1, how much can you increase x_1 and avoid negative variables? Which variable will then become 0?

  (ii) perform a pivot by elementary row operations and solve for the new basic variables.

  (iii) optimal now? If not, go on. (The problem is unbounded, as you see when the basis consists of x_1, x_4.)

  (iv) solve this problem geometrically (possible as $n = 2$).

- After you are comfortable with the method, by hand, you can try out the Vanderbei pivot tool

  https://vanderbei.princeton.edu/JAVA/pivot/advanced.html