Oblig 2 – MAT3100

Deadline 16. April 2025, 14.30

Exercise 1.

Consider the LP problem

$$\begin{array}{lrcrcr} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & & & & & \\ & -x_1 & + & x_2 & \geq & -2 \\ & x_1 & - & x_2 & \leq & 4 \\ & & & x_2 & \geq & 0 \end{array} \tag{1}$$

and note that x_1 is a "free" variable (arbitrary real number).

a)

Explain why problem (1) is equivalent to

$$\begin{array}{lrcrcr} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & & & & & \\ & x_1 & - & x_2 & \leq & 2 \\ & & & x_2 & \geq & 0 \end{array} \tag{2}$$

Hint: Multiply one inequality by ...., and note that one inequality is redundant.

Solution: Multiply the first inequality by -1, and then we see that the second inequality is redundant.

b)

Explain why problem (2) is equivalent to

$$\begin{array}{lrcrcrcr} \max & 2x_1' & & -2x_1'' & + & 3x_2 & & \\ \text{s.t.} & & & & & & & \\ & x_1' & & - & x_1'' & - & x_2 & \leq & 2 \\ & & x_1', x_1'', x_2 & \geq & 0 & & & \end{array} \tag{3}$$

Show that problem (2) is unbounded. Do this directly without the simplex algorithm. Then solve (3) with the simplex algorithm (this problem is in our standard form).

Solution: We replace x_1 by $x_1' - x_1''$ where $x_1'', x_1'' \geq 0$, and using that $x = x_1' - x_1''$. To solve (2) we can increase both x_1 and x_2 so that $x_1 - x_2 = 2$. In fact, let $(x_1, x_2) = (t+2, t)$ for some $t \geq 0$; this is a feasible solution, and by letting t go to infinity, we see that (2) is unbounded. Simplex for problem (3): write down the dictionary with $w = 2 - x_1' + x_2'' + x_2$, and pivot x_1' into the basis, and note then that the problem is unbounded.

c)

Consider the LP:

$$\max\ c^T x,\ \text{subject to}\ \ Ax \le b,$$

where A is an $m \times n$ matrix, and $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. Show that this problem is equivalent to

$$\max\ c^T x^1 - c^T x^2,\ \text{subject to}\ \ Ax^1 - Ax^2 \le b,\ x^1, x^2 \ge 0,$$

which again can be written as

$$\max \begin{bmatrix} c \\ -c \end{bmatrix}^T \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},\ \text{subject to}\ \ [A\ \ -A] \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \le b,\ x^1, x^2 \ge 0,$$

Solution: Similar to above, we replace x by $x^1 - x^2$ where $x^1, x^2 \ge 0$, and using that $x = x^1 - x^2$.

Exercise 2.

Consider the LP given by the dictionary

$$\begin{array}{lrcrcrcr} \zeta = & 10 & - & 7x_1 & & & + & 2x_3 \\ \hline x_4 = & 1 & & & + & 3x_2 & - & 4x_3 \\ x_5 = & 2 & - & x_1 & + & x_2 & & \\ x_6 = & 0 & + & 3x_1 & & & - & x_3 \end{array} \tag{4}$$

a)

Solve the LP using the simplex algorithm; find an optimal solution and corresponding optimal value.

Solution: Make a pivot (degenerate): x_3 into basis and x_6 out. This gives

$$\begin{array}{lrcrcrcr} \zeta = & 10 & - & x_1 & & & - & 2x_6 \\ \hline x_4 = & 1 & - & 12x_1 & + & 3x_2 & + & 4x_6 \\ x_5 = & 2 & - & x_1 & + & x_2 & & \\ x_3 = & 0 & + & 3x_1 & & & - & x_6 \end{array}$$

So: $x_4 = 1$, $x_5 = 2$ and $x_1 = x_2 = x_3 = x_6 = 0$ is optimal. Optimal value is 10.

b)

Find all optimal solutions of (4). Give reasons for your answer.

Solution: The optimal solutions are: $x_4 = 1 + 3x_2$, $x_5 = 2 + x_2$, $x_1 = x_3 = x_6 = 0$ and $x_2 \ge 0$.

This is seen from the optimal dictionary as $\bar{c}_2 = 0$ and for all $x_2 \ge 0$ the basic variables x_4, x_5, x_3 will be nonnegative.

Exercise 3.

Consider the LP

$$\begin{array}{lrcrcrcl} \max & -7x_1 & & & + & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & & - & x_2 & + & x_3 & \leq & 1 \\ & x_1 & + & x_2 & + & x_3 & \leq & 2 \\ & & & x_1, x_2, x_3 \geq 0. & & & & \end{array}$$

a)

Write down the dual problem. Show that $x = (0, 1/2, 3/2)$ is optimal in the primal problem and $y = (1, 1)$ is optimal in the dual problem.

Solution: The dual problem is

$$\begin{array}{lrcrcl} \min & y_1 & + & 2y_2 & & \\ \text{subject to} & & & & & \\ & & & y_2 & \geq & -7 \\ & -y_1 & + & y_2 & \geq & 0 \\ & y_1 & + & y_2 & \geq & 2 \\ & & y_1, y_2 \geq 0. & & & \end{array}$$

We easily check that x satisfies the constraints in the primal, so x is a feasible solution. Similarly, y is a feasible solution of the dual. Moreover, the corresponding values of the objective functions are

$$7x_1 + 2x_3 = 3, \quad \text{and} \quad y_1 + 2y_2 = 3.$$

Since these two values are equal, by weak duality, the two solutions are optimal in their respective problems.