UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3110/MAT4110 — Introduction to numerical analysis
Day of examination:	22 January 2026
Examination hours:	0900 – 1300
This problem set consists of 3 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 Conditioning

Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$ and for any $n \times n$ real matrix A, let

$$\|A\| := \max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

be the associated operator norm of A.

Suppose A is nonsingular. Show that for any $\mathbf{b}, \mathbf{e} \in \mathbb{R}^n$ with $\mathbf{b} \neq 0$

$$\frac{\|\mathbf{e}\|}{\|\mathbf{b}\|} \leq \|A\|\|A^{-1}\| \frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{x}\|},$$

where $\mathbf{x}$ and $\mathbf{y}$ are solutions of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b} + \mathbf{e}$.

Problem 2 Matlab program

Write a Matlab function `[L,U] = mylu(A)` that computes the LU factorization of an $n \times n$ matrix A, assuming that no pivoting is required.

Problem 3 Cholesky

Use the Cholesky factorization or otherwise to determine where the matrix

$$A = \begin{bmatrix} 2 & 8 & -4 \\ 8 & 21 & 7 \\ -4 & 7 & 10 \end{bmatrix}$$

is positive-definite.

(Continued on page 2.)

Problem 4 Minimization

Let A be an $m \times n$ matrix with $m \geq n$ and suppose that A has full rank. Given a vector $\mathbf{b} \in \mathbb{R}^m$, we want to find the unique vector $\mathbf{x} \in \mathbb{R}^n$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|_2$.

4a QR

What is a QR factorization of A and how can we use it to find $\mathbf{x}$?

4b Normal equations

What are the normal equations for the minimization problem? Describe briefly how they are derived.

Problem 5 Matrix norm

Recall that the 2-norm of a real $n \times n$ matrix A is

$$\|A\|_2 := \max_{\mathbf{x} \neq 0} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and

$$\|\mathbf{x}\|_2 := \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

Using the singular value decomposition of A, show that the 2-norm of A equals σ_1, the largest singular value of A.

Problem 6 Polynomial interpolation

The divided difference $f[x_0, x_1, \ldots, x_n]$ of a function f at the distinct points $x_0, x_1, \ldots, x_n$ is the leading coefficient of the polynomial p of degree at most n that interpolates f at $x_0, x_1, \ldots, x_n$. Using the Lagrange form of p, or otherwise, find $f[x_0, x_1, \ldots, x_n]$ as a linear combination of $f(x_0), f(x_1), \ldots, f(x_n)$.

Problem 7 Newton's method

We want to solve the equation

$$f(x) = x^3 - c = 0, \tag{1}$$

for x where c is a given positive real number.

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7a

If $\{x_k\}$ is the sequence generated by Newton's method, show that

$$x_{k+1} = \frac{2x_k^3 + c}{3x_k^2}. \qquad k = 0, 1, 2, \ldots. \tag{2}$$

7b

If $x_* > 0$ is the root of f in (1), and the k-th error is $e_k = x_k - x_*$, find

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2}. \tag{3}$$

What does this say about the rate of convergence?

Problem 8 Gradient descent

Suppose

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

from some positive definite matrix $A \in \mathbb{R}^{n\times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$, Suppose we want to find the unique minimum $\mathbf{x}* \in \mathbb{R}^n$ of F using a descent method. If $\mathbf{x}$ is the current approximation to $\mathbf{x}*$, the next approximation has the form

$$\mathbf{x}' = \mathbf{x} + \omega\mathbf{d},$$

where $\mathbf{d}$ is the search direction.

If

$$\mathbf{g} = \nabla F(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$$

denotes the gradient of F at $\mathbf{x}$, what is the descent condition on $\mathbf{d}$? Assuming the descent condition holds, find $\mathbf{x}'$ by minimizing F in the direction $\mathbf{d}$.

Good luck!