UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3110/MAT4110 — Introduction to numerical analysis
Day of examination:	22 January 2026
Examination hours:	0900 – 1300
This problem set consists of 7 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 Conditioning

Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$ and for any $n \times n$ real matrix A, let

$$\|A\| := \max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

be the associated operator norm of A.

Suppose A is nonsingular. Show that for any $\mathbf{b}, \mathbf{e} \in \mathbb{R}^n$ with $\mathbf{b} \neq 0$

$$\frac{\|\mathbf{e}\|}{\|\mathbf{b}\|} \leq \|A\|\|A^{-1}\| \frac{\|\mathbf{y}-\mathbf{x}\|}{\|\mathbf{x}\|},$$

where $\mathbf{x}$ and $\mathbf{y}$ are solutions of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b} + \mathbf{e}$.

Answer: Subtracting $A\mathbf{x} = \mathbf{b}$ from $A\mathbf{y} = \mathbf{b} + \mathbf{e}$ we find $A(\mathbf{y} - \mathbf{x}) = \mathbf{e}$. Taking norms

$$\|\mathbf{e}\| \leq \|A\|\|\mathbf{y}-\mathbf{x}\|, \quad \|\mathbf{x}\| \leq \|A^{-1}\|\|\mathbf{b}\|.$$

But then $\frac{1}{\|\mathbf{b}\|} \leq \frac{\|A^{-1}\|}{\|\mathbf{x}\|}$, and the result follows.

Problem 2 Matlab program

Write a Matlab function `[L,U] = mylu(A)` that computes the LU factorization of an $n \times n$ matrix A, assuming that no pivoting is required.

Answer:

(Continued on page 2.)

```
function [L,U] = mylu(A)

n = size(A,1);
L=zeros(n,n);
U=zeros(n,n);
B = A;

for k=1:n
  U(k,:) = B(k,:);
  L(:,k) = B(:,k) / B(k,k);
  B = B - L(:,k) * U(k,:);
end
```

Problem 3 Cholesky

Use the Cholesky factorization or otherwise to determine where the matrix

$$A = \begin{bmatrix} 2 & 8 & -4 \\ 8 & 21 & 7 \\ -4 & 7 & 10 \end{bmatrix}$$

is positive-definite.

Answer We can use the Cholesky algorithm to determine whether A is positive definite. By a theorem from the course, A is positive-definite if and only if it has an LDL^T factorization where the diagonal elements of D are positive.

We initialize $A_0 = A$. In the first step we use the first column of A_0:

$$\mathbf{l}_1 = \frac{1}{2}\begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix},$$

and the first diagonal element of A_0:

$$D_{1,1} = 2.$$

Then we find

$$\mathbf{l}_1\mathbf{l}_1^T = \begin{bmatrix} 1 & 4 & -2 \\ 4 & 16 & -8 \\ -2 & -8 & 4 \end{bmatrix},$$

and so

$$D_{1,1}\mathbf{l}_1\mathbf{l}_1^T = \begin{bmatrix} 2 & 8 & -4 \\ 8 & 32 & -16 \\ -4 & -16 & 8 \end{bmatrix},$$

and we let

$$A_1 = A_0 - D_{1,1}\mathbf{l}_1\mathbf{l}_1^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -11 & 23 \\ 0 & 23 & 2 \end{bmatrix}.$$

In the second step of the algorithm we set $D_{2,2}$ to be the second element of the second column of A_1. Thus $D_{2,2} = -1$. Thus A is not positive-definite.

(Continued on page 3.)

Problem 4 Minimization

Let A be an $m \times n$ matrix with $m \geq n$ and suppose that A has full rank. Given a vector $\mathbf{b} \in \mathbb{R}^m$, we want to find the unique vector $\mathbf{x} \in \mathbb{R}^n$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|_2$.

4a QR

What is a QR factorization of A and how can we use it to find $\mathbf{x}$?

Answer: A QR factorization is $A = QR$ where Q is an $m \times m$ orthogonal matrix and R is $m \times n$ and upper triangular. Then

$$\|A\mathbf{x} - \mathbf{b}\| = \|QR\mathbf{x} - \mathbf{b}\| = \|Q^T(QR\mathbf{x} - \mathbf{b})\| = \|R\mathbf{x} - Q^T\mathbf{b}\|.$$

Next we write R in block form,

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix},$$

where R_1 is $n \times n$, upper triangular and non-singular and $R_2 = 0$, and we correspondingly partition Q into blocks

$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix},$$

where Q_1 is $m \times n$. Then

$$\|R\mathbf{x} - Q^T\mathbf{b})\|^2 = \left\|R_1\mathbf{x} - Q_1^T\mathbf{b}\right\|^2 + \left\|R_2\mathbf{x} - Q_2^T\mathbf{b}\right\|^2 = \|R_1\mathbf{x} - Q_1^T\mathbf{b}\|^2 + \|Q_2^T\mathbf{b}\|^2.$$

Thus the solution to the least squares problem is the vector $\mathbf{x} \in \mathbb{R}^n$ that solves

$$R_1\mathbf{x} = Q_1^T\mathbf{b},$$

and the minimum is

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|A\mathbf{x} - \mathbf{b}\| = \|Q_2^T\mathbf{b}\|.$$

4b Normal equations

What are the normal equations for the minimization problem? Describe briefly how they are derived.

Answer: The normal equations are

$$A^TA\mathbf{x} = A^T\mathbf{b}.$$

The are derived by setting the gradient of the quadratic function

$$f(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - \mathbf{b}\|^2$$

to zero.

(Continued on page 4.)

Since

$$f(\mathbf{x}) = \frac{1}{2}\sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right)^2,$$

$$\begin{aligned}\frac{\partial}{\partial x_k}f(\mathbf{x}) &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i\right)a_{ik} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ik}a_{ij}x_j - \sum_{i=1}^{m} a_{ik}b_i \\ &= \sum_{j=1}^{n}(A^TA)_{kj}x_j - (A^T\mathbf{b})_k \\ &= (A^TA\mathbf{x} - A^T\mathbf{b})_k.\end{aligned}$$

Problem 5 Matrix norm

Recall that the 2-norm of a real $n \times n$ matrix A is

$$\|A\|_2 := \max_{\mathbf{x}\neq 0}\frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ and

$$\|\mathbf{x}\|_2 := \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

Using the singular value decomposition of A, show that the 2-norm of A equals σ_1, the largest singular value of A.

Answer: The SVD of A is

$$A = USV^T,$$

where $U, S, V \in \mathbb{R}^{n,n}$, U, V are orthogonal, and S is diagonal with diagonal elements $\sigma_1, \ldots, \sigma_n$, the singular values of A, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Therefore,

$$\|A\|_2 = \max_{\mathbf{x}\neq 0}\frac{\|USV^T\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \max_{\mathbf{x}\neq 0}\frac{\|SV^T\mathbf{x}\|_2}{\|\mathbf{x}\|_2},$$

because U is orthogonal. Letting $\mathbf{x} = V\mathbf{y}$,

$$\|A\|_2 = \max_{V\mathbf{y}\neq 0}\frac{\|S\mathbf{y}\|_2}{\|V\mathbf{y}\|_2} = \max_{\mathbf{y}\neq 0}\frac{\|S\mathbf{y}\|_2}{\|\mathbf{y}\|_2},$$

because V is also orthogonal. Therefore

$$\|A\|_2 = \max_{\mathbf{y}\neq 0}\left(\frac{\sum_{i=1}^{n}\sigma_i^2 y_i^2}{\sum_{i=1}^{n} y_i^2}\right)^{1/2}.$$

Therefore,

$$\|A\|_2 \leq \sigma_1$$

and the inequality is equality because we can choose $\mathbf{y} = (1, 0, \ldots, 0)$.

(Continued on page 5.)

Problem 6 Polynomial interpolation

The divided difference $f[x_0, x_1, \ldots, x_n]$ of a function f at the distinct points $x_0, x_1, \ldots, x_n$ is the leading coefficient of the polynomial p of degree at most n that interpolates f at $x_0, x_1, \ldots, x_n$. Using the Lagrange form of p, or otherwise, find $f[x_0, x_1, \ldots, x_n]$ as a linear combination of $f(x_0), f(x_1), \ldots, f(x_n)$.

Answer: The Lagrange form of p is

$$p(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} f(x_i).$$

Therefore, writing

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots ,$$

we see that the leading coefficient of p is

$$c_n = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{1}{x_i - x_j} f(x_i),$$

which is therefore the desired formula.

Problem 7 Newton's method

We want to solve the equation

$$f(x) = x^3 - c = 0, \tag{1}$$

for x where c is a given positive real number.

7a

If $\{x_k\}$ is the sequence generated by Newton's method, show that

$$x_{k+1} = \frac{2x_k^3 + c}{3x_k^2}. \qquad k = 0, 1, 2, \ldots. \tag{2}$$

Answer: Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 0, 1, 2, \ldots.$$

Since $f(x_k) = x_k^3 - c$ and $f'(x_k) = 3x_k^2$, the result follows.

(Continued on page 6.)

7b

If $x_* > 0$ is the root of f in (1), and the k-th error is $e_k = x_k - x_*$, find

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2}. \tag{3}$$

What does this say about the rate of convergence?

Answer: Since $c = x_*^3$, we find

$$\begin{aligned} e_{k+1} &= \frac{2x_k^3 + x_*^3}{3x_k^2} - x_* \\ &= \frac{1}{3x_k^2}(2x_k^3 + x_*^3 - 3x_k^2 x_*) \\ &= \frac{1}{3x_k^2}(x_k - x_*)^2(x_* + 2x_k) = e_k^2 \frac{x_* + 2x_k}{3x_k^2}. \end{aligned}$$

So

$$\lim_{k\to\infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{x_*}.$$

Since $x_* \neq 0$, $e_{k+1} \approx e_k^2/x_*$ for large k and so the method has quadratic convergence.

Problem 8 Gradient descent

Suppose

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{x}^T\mathbf{b}$$

from some positive definite matrix $A \in \mathbb{R}^{n\times n}$ and vector $\mathbf{b} \in \mathbb{R}^n$, Suppose we want to find the unique minimum $\mathbf{x}* \in \mathbb{R}^n$ of F using a descent method. If $\mathbf{x}$ is the current approximation to $\mathbf{x}*$, the next approximation has the form

$$\mathbf{x}' = \mathbf{x} + \omega\mathbf{d},$$

where $\mathbf{d}$ is the search direction.

If

$$\mathbf{g} = \nabla F(\mathbf{x}) = A\mathbf{x} - \mathbf{b}$$

denotes the gradient of F at $\mathbf{x}$, what is the descent condition on $\mathbf{d}$? Assuming the descent condition holds, find $\mathbf{x}'$ by minimizing F in the direction $\mathbf{d}$.

Answer: The descent condition is that

$$\left[\frac{d}{d\omega}F(\mathbf{x} + \omega\mathbf{d})\right]_{\omega=0} = \mathbf{d}^T\mathbf{g} < 0.$$

(Continued on page 7.)

We minimize the quadratic polynomial $F(\mathbf{x}+\omega\mathbf{d})$ with respect to ω. By the definition of F,

$$\begin{aligned} F(\mathbf{x}+\omega\mathbf{d}) &= \frac{1}{2}(\mathbf{x}+\omega\mathbf{d})^T A(\mathbf{x}+\omega\mathbf{d}) - (\mathbf{x}+\omega\mathbf{d})^T\mathbf{b} \\ &= \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \omega\mathbf{d}^T A\mathbf{x} + \frac{1}{2}\omega^2\mathbf{d}^T A\mathbf{d} - \mathbf{x}^T\mathbf{b} - \omega\mathbf{d}^T\mathbf{b} \\ &= F(\mathbf{x}) + \omega\mathbf{d}^T\mathbf{g} + \frac{1}{2}\omega^2\mathbf{d}^T A\mathbf{d}. \end{aligned}$$

The minimum of the quadratic is attained when

$$\frac{d}{d\omega}F(\mathbf{x}+\omega\mathbf{d}) = 0,$$

i.e., when

$$\mathbf{d}^T\mathbf{g} + \omega\mathbf{d}^T A\mathbf{d} = 0,$$

which implies that

$$\omega = -\frac{\mathbf{d}^T\mathbf{g}}{\mathbf{d}^T A\mathbf{d}},$$

which is positive if the descent condition holds. Thus

$$\mathbf{x}' = \mathbf{x} - \frac{\mathbf{d}^T\mathbf{g}}{\mathbf{d}^T A\mathbf{d}}\mathbf{d}.$$

Good luck!