The LU factorization of a Vandermonde matrix

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Let A be the $n \times n$ Vandermonde matrix

$$A := [x_i^{j-1}]_{i,j=1,2,\ldots,n},$$

where $x_1, \ldots, x_n \in \mathbb{R}$ are distinct. The cases $n = 2$ and $n = 3$ are

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}.$$

A variant of the LU factorization we studied in the lecture scales the rows of U so that U has ones on the diagonal instead of L. With this variant of LU the factorization of A when $n = 2$ is

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 0 & 1 \end{bmatrix}.$$

What is the LU factorization of A in the case $n = 3$ (where U again has ones on the diagonal)? What is the determinant of A?