Lecture 7: Polynomial interpolation and Chebyshev polynomials

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These notes are based on Section 3.1 of the book.

1 Lagrange interpolation

Interpolation describes the problem of finding a function that passes through a given set of values $f_0, f_1, \ldots, f_n$ at data points $x_0, x_1, \ldots, x_n$. Such a function is called an interpolant. If p is an interpolant then $p(x_i) = f_i$, $i = 0, 1, \ldots, n$. Let us suppose that the points and values are in $\mathbb{R}$. If the values f_i are samples $f(x_i)$ from some real function f then p approximates f.

One choice of p is a polynomial of degree at most n. Using the Lagrange basis functions

$$L_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{x - x_j}{x_k - x_j}, \qquad k = 0, 1, \ldots, n,$$

we can express p explicitly as

$$p(x) = \sum_{k=0}^{n} f_k L_k(x).$$

Because $L_k(x_i) = 0$ when $i \neq k$ and $L_k(x_k) = 1$, it is easy to check that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$. The interpolant p is also unique among all polynomial interpolants of degree $\leq n$. This is because if q is another such interpolant, the difference $r = p - q$ would be a polynomial of degree $\leq n$

such that $r(x_i) = 0$ for $i = 0, 1, \ldots, n$. Thus r would have at least $n+1$ zeros and by a result from algebra this would imply that $r = 0$.

For this kind of interpolation there is an error formula.

Theorem 1 *Suppose $f \in C^{n+1}[a, b]$ and let $f_i = f(x_i)$, $i = 0, 1, \ldots, n$, where $x_0, x_1, \ldots, x_n$ are distinct points in $[a, b]$. Let p be the polynomial interpolant to f of degree $\leq n$ and let $e = f - p$ be the error. For each $x \in [a, b]$, there exists $\xi \in (a, b)$ such that*

$$e(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega(x),$$

where

$$\omega(x) := \prod_{i=0}^{n}(x - x_i).$$

Proof. The formula is true if $x = x_j$ for any $j \in \{0, 1, \ldots, n\}$ with any choice of ξ. So let x be any other point in $[a, b]$, and define the function

$$\phi(t) := e(t)\omega(x) - e(x)\omega(t), \qquad t \in [a, b].$$

Then, $\phi(x_j) = 0$, $j = 0, 1, \ldots, n$, and $\phi(x) = 0$. Therefore, ϕ has at least $n + 2$ zeros in $[a, b]$. By Rolle's theorem, ϕ' has at least one zero between each consecutive pair of zeros of ϕ, and so ϕ' has at least $n + 1$ zeros in (a, b). We continue in this way: $\phi^{(r)}$ has at least $n + 2 - r$ zeros in $[a, b]$ for $r = 1, 2, \ldots, n + 1$. Thus there is at least one point ξ in (a, b) such that $\phi^{(n+1)}(\xi) = 0$. The $(n + 1)$-st derivative of ϕ is

$$\phi^{(n+1)}(t) = f^{(n+1)}(t)\omega(x) - (n + 1)!e(x), \qquad t \in [a, b].$$

Hence

$$f^{(n+1)}(\xi)\omega(x) - (n + 1)!e(x) = 0.$$

□

2 Asymptotic approximation

Let us keep $[a, b]$ fixed and consider two kinds of asymptotic approximation:

1. Fix n and let $h = x_n - x_0 \to 0$, where $x_0 < x_1 < \cdots < x_n$.

2. Let $n \to \infty$.

Consider the first of these. Let

$$\|g\|_{[a,b]} = \max_{a \le x \le b} |g(x)|.$$

From the theorem we have

$$|e(x)| \le \frac{\|f^{(n+1)}\|_{[a,b]}}{(n+1)!} h^{n+1}, \qquad x \in [x_0, x_n].$$

Thus,

$$\|e\|_{[x_0,x_n]} = O(h^{n+1}) \quad \text{as } h \to 0.$$

It is this rate of approximation that is important when we use polynomial interpolation to construct composite numerical quadrature rules and local numerical differentiation formulas.

For the second kind of asymptotic approximation, we want to try to reduce the error e on the whole interval $[a, b]$ by increasing n. In this case, the distribution of the points $x_0, x_1, \ldots, x_n$ plays an important role. A classical example of the importance of this distribution comes from Runge's function

$$f(x) = \frac{1}{1 + 25x^2}, \quad x \in [-1, 1].$$

If we use the equally spaced points

$$x_i = -1 + 2i/n, \quad i = 0, 1, \ldots, n,$$

the error $\|f - p_n\|_{[-1,1]}$ of the interpolant $p = p_n$ gets larger as n increases, and in fact diverges as $n \to \infty$. This is known as Runge's phenomenon.

A solution to this problem is to distribute the points $x_0, x_1, \ldots, x_n$ in $[-1, 1]$ in a different way, if we are free to choose them. With the error formula of Theorem 1 in mind, the solution is to choose the distribution that minimizes $\|\omega\|_{[-1,1]}$.

3 Chebyshev polynomials

As we will see, we minimize $\|\omega\|_{[-1,1]}$ by letting the x_i be the Chebyshev points

$$x_i = \cos\left(\frac{\pi}{2} \frac{2i+1}{n+1}\right), \quad i = 0, 1, \ldots, n. \tag{1}$$

To explain this we need to define the Chebyshev polynomials. The Chebyshev polynomial of degree $n \geq 0$ is

$$T_n(x) = \cos(n \arccos(x)).$$

Clearly,

$$T_0(x) = 1, \qquad T_1(x) = x,$$

and one can derive the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \qquad n \geq 1.$$

Therefore,

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x,$$

and so on. From the definition of T_n we see that the points (1) are the $n+1$ zeros of T_{n+1}. Moreover, the extrema of the Chebyshev polynomials in $[-1, 1]$ are -1 or 1 and

$$T_{n+1}(1) = 1, \quad T_{n+1}(-1) = (-1)^{n+1}.$$

Lemma 1 *The minimal value of $\|\omega\|_{[-1,1]}$ is attained by*

$$\omega = 2^{-n}T_{n+1}.$$

It follows from this lemma that with the Chebyshev points x_i of (1),

$$\|e\|_{[-1,1]} \leq 2^{-n} \frac{\|f^{(n+1)}\|_{[-1,1]}}{(n+1)!}.$$

So if, for example, $f(x) = e^x$ or $f(x) = \cos(x)$, we will have convergence:

$$\|e\|_{[-1,1]} \leq C2^{-n} = O(2^{-n}) \quad \text{as } n \to \infty,$$

and the convergence is exponential.

Proof. The polynomial ω has degree $n+1$ and leading coefficient 1. From the recurrence relation, the leading coefficient of T_{n+1} is 2^n and thus $2^{-n}T_{n+1}$ also has leading coefficient 1.

Now suppose, for the sake of contradiction, that there exists some polynomial p of degree $n+1$ with leading coefficient 1 such that $\|p\|_{[-1,1]} = c$ for some $c < 2^{-n}$. Let r be the polynomial

$$r(x) = 2^{-n}T_{n+1}(x) - p(x),$$

which is a polynomial of degree at most n.

At a point $x = \cos(2k\pi/(n+1))$, $0 \le 2k \le n+1$, we have $T_{n+1}(x) = 1$ and so

$$r(x) = 2^{-n} - p(x) > 0.$$

At a point $x = \cos((2k+1)\pi/(n+1))$, $0 \le 2k \le n+1$, we have $T_{n+1}(x) = -1$ and so

$$r(x) = -2^{-n} - p(x) < 0.$$

Therefore, r changes sign between the $n+2$ points $x = \cos(k\pi/(n+1))$, $0 \le k \le n+1$. Thus, r has at least $n+1$ roots, and so $r = 0$, which is a contradiction. □

4 Exercises

Exercise 3.1 *From the book.*

Exercise 3.2 *From the book.*

Exercise 3 *Derive the recurrence relation for the Chebyshev polynomials.*