Matrix norms

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These notes support the chapter on matrix norms.
For a matrix $A \in \mathbb{C}^{m,n}$, we define, for $p \geq 1$,

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

Theorem 1

$$\|A\|_1 = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|.$$

This is the 'maximal column sum' formula.

Proof. Let

$$K = \max_{j=1,\ldots,n} \sum_{i=1}^{m} |a_{ij}|.$$

Then, for any $x \in \mathbb{C}$,

$$\|Ax\|_1 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}||x_j| = \sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}||x_j| \leq K\|x\|_1.$$

Therefore, $\|A\|_1 \leq K$.

On the other hand, there is some r such that

$$K = \sum_{i=1}^{m} |a_{ir}|,$$

i.e., column r has maximal ℓ_1 norm among the columns of A. Let $x = e_r$. Then,

$$\|Ax\|_1 = \sum_{i=1}^{m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| = \sum_{i=1}^{m} |a_{ir}| = K.$$

Therefore, $\|A\|_1 \geq K$, because $\|x\|_1 = 1$. □

Theorem 2

$$\|A\|_\infty = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|.$$

This is the 'maximal row sum' formula.

The proof is similar.

Proof. Let

$$K = \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}|.$$

Then, for any $x \in \mathbb{C}$,

$$\|Ax\|_\infty = \max_{i=1,\ldots,m} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \max_{i=1,\ldots,m} \sum_{j=1}^{n} |a_{ij}||x_j| \leq K\|x\|_\infty.$$

Therefore, $\|A\|_\infty \leq K$.

On the other hand, there is some r such that

$$K = \sum_{j=1}^{n} |a_{rj}|,$$

i.e., row r has maximal ℓ_1 norm among the rows of A. Let

$$x = [\mathrm{sgn}(a_{r1}), \ldots, \mathrm{sgn}(a_{rn})]^T,$$

where

$$\mathrm{sgn}(a) := \begin{cases} 1 & a > 0; \\ 0 & a = 0; \\ -1 & a < 0. \end{cases}$$

Then,

$$\|Ax\|_\infty \geq \left| \sum_{j=1}^{n} a_{rj} x_j \right| = \left| \sum_{j=1}^{n} a_{rj} \mathrm{sgn}(a_{rj}) \right| = K.$$

Therefore, $\|A\|_\infty \geq K$, because $\|x\|_\infty = 1$ (unless $A = 0$). □