

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3110/MAT4110 — Introduction to numerical analysis

Day of examination: 4 December 2024

Examination hours: 0900–1300

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 Back substitution

Let $A = [a_{ij}]$ be a real $n \times n$ matrix with the property that $a_{ij} = 0$ whenever $j < i$ or $j > i + 1$.

1a

When is A non-singular?

Answer: The condition that $a_{ij} = 0$ for $j < i$ means that A is upper triangular and so A is non-singular if and only if $a_{ii} \neq 0$ for all $i = 1, \dots, n$.

1b

Recall that a flop is an addition, subtraction, multiplication, or division. Assuming that $n \geq 2$ and that A is non-singular, exactly how many flops are necessary to compute the solution $\mathbf{x} \in \mathbb{R}^n$ to the linear system $A\mathbf{x} = \mathbf{b}$, given $\mathbf{b} \in \mathbb{R}^n$?

Answer: We use back substitution:

$$\begin{aligned}x_n &= \frac{1}{a_{n,n}}b_n, \\x_{n-1} &= \frac{1}{a_{n-1,n-1}}(b_{n-1} - a_{n-1,n}x_n), \\&\dots \\x_1 &= \frac{1}{a_{1,1}}(b_1 - a_{1,2}x_2).\end{aligned}$$

Thus there are $1 + 3(n - 1) = 3n - 2$ flops required.

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1c

Write a Matlab (or python) routine `function x=backsolve(A,b)` that returns the solution $\mathbf{x}$ to the linear system $A\mathbf{x} = \mathbf{b}$.

Answer:

```
function x=backsolve(A,b)
n=length(b); x=b;
x(n)=b(n)/A(n,n);
for k=n-1:-1:1
    x(k)=(b(k)-A(k,k+1)*x(k+1))/A(k,k);
end
```

Problem 2 LU

Find the LU factorization of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 6 & 5 \end{bmatrix}.$$

Answer Let

$$u_1^T = \begin{bmatrix} 2 & -1 \end{bmatrix}, \quad l_1 = \frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Then

$$l_1 u_1^T = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix},$$

and

$$A_1 := A - l_1 u_1^T = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}.$$

Now we set

$$u_2^T = \begin{bmatrix} 0 & 8 \end{bmatrix}, \quad l_2 = \frac{1}{8} \begin{bmatrix} 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So $A = LU$ where

$$L = \begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 8 \end{bmatrix}.$$

Problem 3 Polynomial interpolation

Suppose we approximate a function $f \in C^2[-1, 1]$ by the linear polynomial $p(x)$ that interpolates f at two distinct points $x_0, x_1 \in [-1, 1]$. Then there is some $\xi \in (-1, 1)$ such that

$$f(x) - p(x) = \omega(x) \frac{f''(\xi)}{2!}, \quad x \in [-1, 1],$$

(Continued on page 3.)

where

$$\omega(x) := (x - x_0)(x - x_1).$$

Suppose we decide to set $x_0 = -\lambda$ and $x_1 = \lambda$ for some $\lambda \in (0, 1)$. Which value of λ minimizes

$$\|\omega\| := \max_{-1 \leq x \leq 1} |\omega(x)|,$$

and what is the value of $\|\omega\|$ in this case?

Answer: With this choice of the x_i ,

$$\omega(x) = (x - (-\lambda))(x - \lambda) = x^2 - \lambda^2.$$

Since ω is convex, its maximum is attained at $x = -1$ or $x = 1$. So its maximum is $1 - \lambda^2$. Its minimum occurs when $\omega'(x) = 0$, i.e., when $x = 0$, and so its minimum is $-\lambda^2$. So we minimize $\|\omega\|$ by choosing λ such that

$$1 - \lambda^2 = \lambda^2,$$

and so $\lambda = 1/\sqrt{2}$. Then $\|\omega\| = 1/2$.

Problem 4 Square root

Your calculator is broken, the internet is down, and you need to compute the square root of a positive number a . You decide to do this by solving the equation

$$f(x) := x^2 - a = 0,$$

and use Newton's method to approximate the root of f (which is the square root of a).

4a

If $\{x_k\}$, $k = 0, 1, 2, \dots$, is the sequence of approximations in Newton's method, show that x_{k+1} can be expressed as a function of x_k and the ratio a/x_k .

Answer: Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \dots$$

Since $f(x_k) = x_k^2 - a$ and $f'(x_k) = 2x_k$, we have

$$x_{k+1} = x_k - \frac{x_k^2 - a}{2x_k} = \frac{1}{2} \left(x_k + \frac{a}{x_k} \right).$$

(Continued on page 4.)

4b

You need to compute the square root of 3. Starting with $x_0 = 2$ what are your approximations x_1 and x_2 ?

Answer:

$$x_1 = \frac{1}{2} \left(2 + \frac{3}{2} \right) = \frac{7}{4}.$$

$$x_2 = \frac{1}{2} \left(\frac{7}{4} + \frac{3}{7/4} \right) = \frac{1}{2} \left(\frac{7}{4} + \frac{12}{7} \right) = \frac{97}{56}.$$

4c

Let $x_* > 0$ be the root of f and define the k -th error, $e_k := x_k - x_*$. What is the ratio

$$\frac{e_{k+1}}{e_k^2},$$

and what does this say about the rate of convergence?

Answer: Since $a = x_*^2$, we find

$$\begin{aligned} e_{k+1} = x_{k+1} - x_* &= \frac{1}{2} \left(x_k + \frac{x_*^2}{x_k} \right) - x_* \\ &= \frac{1}{2x_k} (x_k^2 + x_*^2 - 2x_k x_*) \\ &= \frac{1}{2x_k} (x_k - x_*)^2 = \frac{e_k^2}{2x_k}, \end{aligned}$$

and so $e_{k+1}/e_k^2 = 1/(2x_k)$. Since $1/(2x_k)$ approaches the constant $1/(2x_*)$ as $k \rightarrow \infty$, the convergence is quadratic.

Problem 5 Least squares

Suppose you have data (t_j, y_j) , $j = 1, \dots, m$, with $t_1, \dots, t_m$ distinct and you want to model it with a function $p(t)$ using least squares approximation, i.e., by minimizing

$$\frac{1}{2} \sum_{j=1}^m (p(t_j) - y_j)^2.$$

5a

You decide to choose a model of the form

$$p(t) := x_1 + x_2 t,$$

with parameters x_1, x_2 . This means minimizing

$$\frac{1}{2} \|A\mathbf{x} - \mathbf{y}\|^2,$$

(Continued on page 5.)

where $\mathbf{x} = [x_1, x_2]^T$ and $\mathbf{y} = [y_1, \dots, y_m]^T$. (i) what is the matrix A , (ii) what are the normal equations, and (iii) what is the QR method for finding the minimum?

Answer: (i) A is the matrix

$$A = \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}.$$

(ii) The normal equations are

$$A^T A \mathbf{x} = A^T \mathbf{y}.$$

(iii) The QR method is to factorize A as $A = Q_1 R_1$, where Q_1 is $m \times 2$ with (two) orthonormal columns, and R_1 is 2×2 and upper triangular. The solution is then the solution to the 2×2 linear system

$$R_1 \mathbf{x} = Q_1^T \mathbf{y},$$

which can be solved by back substitution.

5b

Your research colleague suggests that a better model for your data set would be a function of the form

$$p(t) := \frac{x_2 t}{x_1 + t}.$$

In this case, which two methods could you try in order to minimize

$$\frac{1}{2} \sum_{j=1}^m (p(t_j) - y_j)^2,$$

based on Newton's root-finding method?

Answer: We want to minimize the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$f(\mathbf{x}) := \frac{1}{2} \sum_{j=1}^m r_j^2,$$

where

$$r_j := \frac{x_2 t_j}{x_1 + t_j} - y_j.$$

To do this we can try to find $\mathbf{x} = [x_1, x_2]^T$ where $\nabla f(\mathbf{x}) = 0$. Newton's method for this is the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - (H(f(\mathbf{x}^{(k)})))^{-1} \nabla f(\mathbf{x}^{(k)}), \quad k = 0, 1, 2, \dots,$$

where

$$\nabla f(\mathbf{x}) := \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}, \quad H(f(\mathbf{x})) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} \end{bmatrix}.$$

(Continued on page 6.)

The other iterative method is the Gauss-Newton method. To explain this let $\mathbf{r} := [r_1, \dots, r_m]^T$ and

$$J_r(\mathbf{x}) := \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} \\ \vdots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \frac{\partial r_m}{\partial x_2} \end{bmatrix}.$$

Then

$$\nabla f = J_r \mathbf{r},$$

and

$$H(f) = J_r^T J_r + Q,$$

where

$$Q := \sum_{i=1}^m r_i H(r_i).$$

The Gauss-Newton method is the approximation to Newton's method made by approximating $H(f)$ by $J_r^T J_r$.

Good luck!