UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i:	MAT2250 — Discrete Mathematics
Eksamensdag:	Onsdag 14. juni 2023
Tid for eksamen:	15.00 – 19.00
Oppgavesettet er på 11 sider.	
Vedlegg:	Ingen
Tillatte hjelpemidler:	Ingen.

Kontroller at oppgavesettet er komplett før
du begynner å besvare spørsmålene.

(Fortsettes på side 2.)

Important rule:

You need to provide sufficient arguments/explanations (for example, citing a relevant theorem, showing supporting calculations) for all claims in order to receive full points. A claim without any support may receive no point.

Notations:

The following notations, from the course's lectures, are repeated here for your convenience.

An Euler circuit in a multigraph with q edges is a sequence of edges $k_1, \ldots, k_q$ with k_i incident to k_{i+1} for $i = 1, \ldots, q-1$, and k_q is incident to k_1. If we do not require that k_q is incident to k_1, then we have an open Euler circuit.

In a weighted graph, the weight of a path or a circuit is the sum of weights of the edges in the path or the circuit.

A Hamiltonian circuit of a graph with n vertices is a circuit of length n in which each vertex appears exactly once.

A spanning tree for a connected graph G is a subgraph T of G which is a tree so that between any pair of distinct vertices x and y of G there is a path in T connecting x and y.

A rooted tree is a tree with a distinct vertex called the root. Vertices in the tree having no successors are leaves. The length of a vertex is the length of the unique path from the root to that vertex.

An (n, q) tree is a rooted tree with n leaves, where any vertex has at most q immediate successors.

Kraft's inequality: Given integers $L_1, \ldots, L_n \geq 1$ and a positive integer $q \geq 2$. If $\sum_{i=1}^{n} q^{-L_i} \leq 1$, then there is an (n, q)-tree T whose leaves have lengths $L_1, \ldots, L_n$.

Huffman's algorithm to construct an optimal (n, q)-tree with preassigned probabilities on the leaves consists of two steps. Step 1: we start from the n leaves with the corresponding probabilities. While there are still more than 1 leaf, repeatedly replace the q leaves having smallest probabilities by a new leaf whose probability is the sum of the probabilities of the q replaced leaves. Step 2 (expanding step): from the trivial tree, whose only vertex has probability 1, expand using the information from Step 1 to obtain an (n,q) -tree whose leaves have the given probabilities.

For a directed edge k in a graph which goes from vertex x to vertex y, we define $k^+ = y$ and $k^- = x$.

An admissible flow in a network is a function $f : E \to \mathbf{N}_0$, from the set of edges E to the set of non-negative integers, which satisfies the following two conditions: 1) for all edge k we have $0 \leq f(k) \leq c(k)$, where $c(k)$ is the capacity of the edge k, and 2) for every vertex x (which is not the source s or the sink t) we have $\partial f(x) = 0$, where $\partial f(x) := \sum_{k \in E:\ k^+ = x} f(k) - \sum_{k \in E:\ k^- = x} f(k)$ is the net flow at the vertex x.

The value of an admissible flow in a network is given by the formula $w(f) := \partial f(t)$, the net flow at the sink t.

An augmenting path (or increasing path) in a network with a given admissible flow f is an undirected path P: $s = x_0, x_1, x_2, \ldots, x_t = x$ from s to x which satisfies the following two properties: 1) $f(k_i) < c(k_i)$ for every "forward edge"$k_i = (x_{i-1}, x_i)$. 2) $0 < f(k_j)$ for every "backward edge"$k_j = (x_j, x_{j-1})$.

(Fortsettes på side 3.)

The O-notation: $a_n = O(b_n)$ if there is a constant $C > 0$ so that for all large enough n we have $|a_n| \leq C|b_n|$.

Partial fractional decomposition: Let $f(t) = P(t)/Q(t)$ where $P(t)$ and $Q(t)$ are polynomials in t. Assume that $Q(t) = c(t-\alpha_1)^{n_1}(t-\alpha_2)^{n_2}$, where c, α_1, α_2 are constants, and $n_1, n_2 \geq 1$ are positive integers. Then the partial fractional decomposition of $f(t)$ is the following expression:

$$f(t) = H(t) + \sum_{i=1}^{n_1} \frac{a_i}{(t-\alpha_1)^i} + \sum_{j=1}^{n_2} \frac{b_j}{(t-\alpha_2)^j}.$$

Here $H(t)$ is a polynomial, a_i and b_j are constants.

(Fortsettes på side 4.)

Oppgave 1 (vekt 25%)

A weighted graph G has 6 vertices H,I,J,K,L,M. The following table Tabell 1 gives a summary of the graph G, where a blank entry means that there is no edge between the two vertices, and a number x means there is an edge between the two vertices with weight x. For example, there is an edge between the vertices J and K, with weight 8.

Tabell 1: Weights of the edges in the graph G

	H	I	J	K	L	M
H		4	2	6	9	
I	4		1	3	6	3
J	2	1		8		7
K	6	3	8		8	3
L	9	6		8		6
M		3	7	3	6	

a (vekt 5%)

Is the graph G connected? What is the adjacency matrix A of the graph G?

PROOF: The adjacency matrix A is the following:

Tabell 2: The incidence matrix of the graph G

	H	I	J	K	L	M
H	0	1	1	1	1	0
I	1	0	1	1	1	1
J	1	1	0	1	0	1
K	1	1	1	0	1	1
L	1	1	0	1	0	1
M	0	1	1	1	1	0

The graph G is connected. Indeed, we have a path H, I, J, K, M, L in the graph.

b (vekt 5%)

Compute A^2, and deduce the number of paths of length 2 between the vertices I and K, and between the vertices J and M.

PROOF: The square of A is:

(Fortsettes på side 5.)

Tabell 3: The incidence matrix of the graph G

	H	I	J	K	L	M
H	4	3	2	3	2	4
I	3	5	3	4	3	3
J	2	3	4	3	4	2
K	3	4	3	5	3	3
L	2	3	4	3	4	2
M	4	3	2	3	2	4

Hence, between I and K, there are 4 paths of length 2. Similarly, between J and M, there are 2 paths of length 2.

c (vekt 5%)

Does the graph G have an Euler circuit or open Euler circuit? Explain why. In case it has an Euler circuit or open Euler circuit, show step by step how you construct an Euler circuit or open Euler circuit for the graph G.

PROOF: The degrees of the vertices in the graph G are:
H has degree 4 (an even number)
I has degree 5 (an odd number)
J has degree 4 (an even number)
K has degree 5 (an odd number)
L has degree 4 (an even number)
M has degree 4 (an even number)

Since G has exactly 2 vertices of odd degrees, G has no Euler circuit, but it has an open Euler circuit. Such an open Euler circuit will start from a vertex of odd order and end at a vertex of odd order. For example: $I, H, J, I, M, J, K, I, L, H, K, L, M, K$.

d (vekt 5%)

The following table Tabell 2 represents a weighted complete graph G_1 having G as a subgraph:

Tabell 4: Weights of the edges in the graph G_1

	H	I	J	K	L	M
H		4	2	6	9	7
I	4		1	3	6	3
J	2	1		8	5	7
K	6	3	8		8	3
L	9	6	5	8		6
M	7	3	7	3	6	

Find a spanning tree T for the graph G_1. Double all edges of T to obtain a graph T'. Describe T, T'.

PROOF: A spanning tree of G is for example $T = H, I, J, K, L, M$. The double tree T' is $H, I, J, K, L, M, L, K, J, I, H$.

(Fortsettes på side 6.)

e (vekt 5%)

Find an Euler circuit of T', and use it to construct a Hamiltonian circuit for the graph G_1. What is the weight of the constructed Hamiltonian circuit?

PROOF: The Euler circuit is as above. Now, we need to, starting from the point H, go along the edges in this circuit, and then delete vertices already appeared to get a Hamiltonian circuit. Hence, the Hamiltonian circuit is H, I, J, K, L, M, H. The weight of the Hamiltonian circuit is the sum of the weights of the edges HI, IJ, JK, KL, LM, MH. The answer is $4 + 1 + 8 + 8 + 6 + 7$ and is 34.

Oppgave 2 (vekt 25%)

Given the numbers $p_1 = 0.2$, $p_2 = 0.15$, $p_3 = 0.1$, $p_4 = 0.05$, $p_5 = 0.25$, $p_6 = 0.25$.

Also, given the numbers $l_1 = 1$, $l_2 = 1$, $l_3 = 5$, $l_4 = 6$, $l_5 = 6$, $l_6 = 3$.

Let $q \geq 2$ be an integer. Let $\mathcal{T}_q$ be the set of $(6, q)$-trees with leaves $x_1, \ldots, x_6$. If $T \in \mathcal{T}_q$, where the length of the leaf x_i is $l(x_i)$, we define

$$\overline{L}_q(T) := \sum_{i=1}^{6} p_i l(x_i).$$

We also define

$$\overline{L}_q := \min_{T \in \mathcal{T}_q} \overline{L}_q(T).$$

a (vekt 5%)

Is there a $(6, 4)$-tree T with leaves $x_1, \ldots, x_6$, where the leaf x_i has length l_i? Explain.

PROOF: We use Kraft's inequality. We check whether $\sum_{i=1}^{6} 4^{-L_i}$ and 1. We show that indeed the sum is < 1. We have $4^{-l_i} \leq 4^{-2}$ if $l_i \geq 2$. Hence, the sum is

$$< 4^{-1} + 4^{-1} + 4^{-2} + 4^{-2} + 4^{-2} + 4^{-2} < 1.$$

Hence, there is a $(6, 4)$ tree with these preassigned lengths for the leaf.

b (vekt 10%)

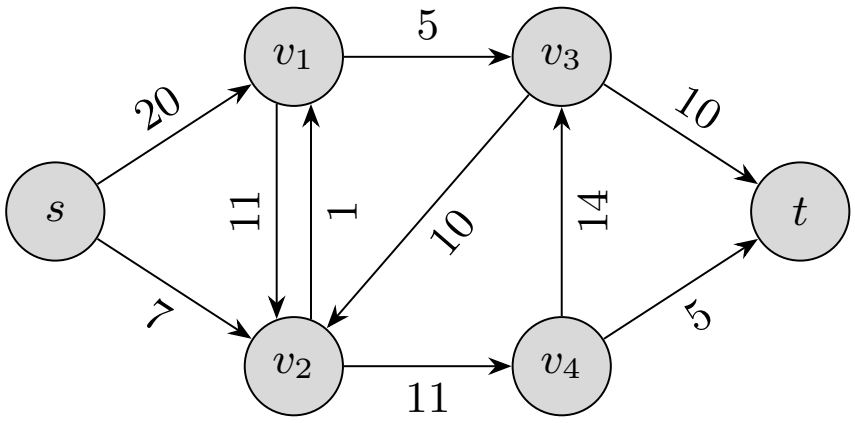
Given integers $L_1, \ldots, L_6 \geq 1$. Show that if $q \geq 6$, then there is a $(6, q)$-tree T with leaves $x_1, \ldots, x_6$ so that $l(x_i) = L_i$, where $l(x_i)$ is the length of x_i.

PROOF: By Kraft's inequality, it suffices to show that $\sum_{i=1}^{6} q^{-L_i} \leq 1$. Since $q \geq 6$ and $L_i \geq 1$, we have $q^{-L_i} \leq 1/6$ for all i. Therefore, $\sum_{i=1}^{6} q^{-L_i} \leq \sum_{i=1}^{6} 1/6 = 6 \times (1/6) = 1$.

c (vekt 10%)

Using Huffman's algorithm, compute $\overline{L}_3 := \min_{T \in \mathcal{T}_3} \overline{L}_3(T)$.

(Fortsettes på side 7.)



Figur 1: A network, with capacities.

PROOF: Huffman's algorithm works by collecting, recursively q leaves with smallest weights into a new leaf. At the end, it reduces to the trivial tree, and from that expands again to get the desired tree.

We need $q-1|n-1$, hence we add one more leaf (with weight 0) in. The weights, in the increasing order, are 0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.25.

We combine the smallest 3 weights into one, hence get $0.15 = 0 + 0.05 + 0.1, 0.15, 0.2, 0.25, 0.25$. Similarly, next step we get 0.25, 0.25, 0.5. Then we come to the trivial tree, whose unique vertex has probability 1.

We now expand from this, using the information from before, to get the following tree:

The root r, with probability 1.

Length 1: 3 children of the roots, with probabilities 0.25, 0.25 and 0.5. Where the two with probability 0.25 are leaves.

Length 2: 3 children of the vertex with probability 0.5, whose probabilities are 0.15, 0.15, 0.2. One vertex with probability 0.15 and the vertex with probability 0.2 are leaves.

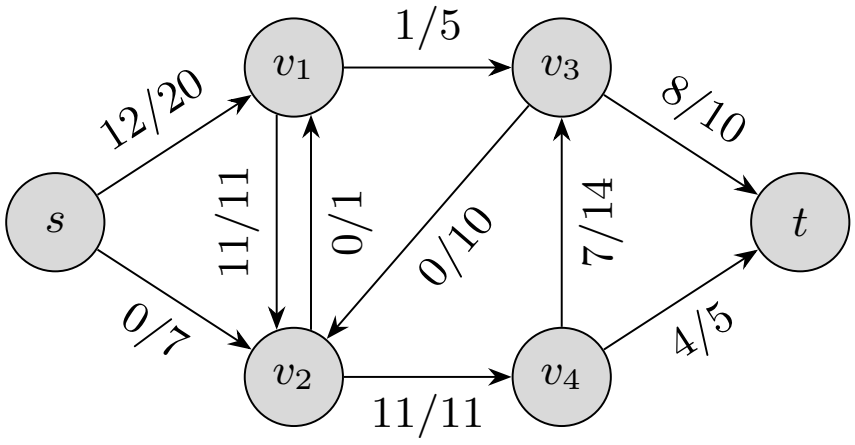
Length 3: 3 children of the vertex with probability 0.15, whose probabilities are 0, 0.05 and 0.1. All of them are leaves. We can delete the leaf with probability 0.

Hence, $\overline{L}_3 = 1 \times 0.25 + 1 \times 0.25 + 2 \times 0.2 + 2 \times 0.15 + 3 \times 0.05 + 3 \times 0.1 = 1.65$.

Oppgave 3 (vekt 25%)

Let a network be given by the following diagram Figur 1. Here, the network has s = source vertex, t = sink vertex, and 4 other vertices v_1, v_2, v_3, v_4. The number on a directed edge means the capacity of the edge. If f is an admissible flow on the network, we denote the value of f by $w(f)$.

(Fortsettes på side 8.)



Figur 2: An admissible flow f_0 on the network in Figur 1.

a (vekt 5%)

Figur 2 below gives an admissible flow f_0 on this network. Here a pair a/b over an edge means that the capacity of the edge is b, and the value of the flow on the edge is a.

Show that $P : s = x_0, v_2, v_1, v_3, v_4, t$ is an augmenting path from s to t for the flow f_0.

PROOF: We need to check that for any forward edge, then $b > a$, and for any backward edge then $a > 0$. In the path P:

s, v_2 is a forward edge with the pair $a/b = 0/7$, and $7 > 0$ as wanted. $b - a = 7$.

v_2, v_1: There are two paths connecting v_2, v_1. The forward path has the pair $0/1$ and $1 > 0$ as wanted. $b - a = 1$. (The backward path has the pair $11/11$, and is also legit since $11 > 0$.)

v_1, v_3: This is the forward path, with the pair $1/5$, and $5 > 1$ as wanted. $b - a = 4$.

v_3, v_4: this is a backward path, with the pair $7/14$, and $7 > 0$ as wanted. $a = 7$.

v_4, t: this is a forward path, with the pair $4/5$, and $5 > 4$ as wanted. $b - a = 1$.

So the path P is an augmenting path.

b (vekt 10%)

From the path P, construct a new admissible flow f_1, different from f_0 only on the path P, for which $w(f_1) > w(f_0)$. What is the value $w(f_1)$?

PROOF: We first find the number ϵ, which is the minimum of all the values $b - a$ of forward paths and of all the values a for backward paths. From the above question, we find that $\epsilon = 1$.

Now we add to the value a of each forward path in P the value ϵ, and subtract ϵ from the value a of each backward path.

(Fortsettes på side 9.)

For the edge s, v_2: the pair a/b is now $1/7$.
For the edge v_2, v_1: the pair a/b is $1/1$.
For the edge v_1, v_3: the pair a/b is $2/5$.
For the path v_3, v_4: the pair a/b is $6/14$.
For the path v_4, t: the pair a/b is $5/5$.
Then the value of $w(f_1)$ is the net flow at the sink t, which is $5+8=13$.

c (vekt 10%)

Show that there is another augmenting path P' (different from the path P in Subproblem a) from s to t for the flow f_0. From this, construct a new admissible flow f_2 (different from f_0 only on the path P') for which $w(f_2) > w(f_0)$. What is the value $w(f_2)$?

PROOF: Consider the path $P' = s, v_1, v_3, t$. All the 3 edges are forward edges with $b > a$. Hence it is an augmenting path. The minimum of $b-a$ on these paths is $\epsilon = 10-8 = 2$. We add ϵ to the a value of all these 3 paths. In particular, the a/b pair for the path v_3, t is now $10/10$. The value of this admissible flow is $w(f_2) = 10+4 = 14$.

Oppgave 4 (vekt 25%)

a (vekt 5%)

Let α be a constant. Assume that the power series for the function

$$\frac{1}{(1+\alpha t)^2}$$

is

$$\sum_{n=0}^{\infty} a_n t^n.$$

Show that if $0 \leq \alpha \leq 1$ then $a_n = O(n)$.

Hint: To find a closed form formula for a_n, use geometric series, and that the derivative of a power series of a function is the power series of the derivative of the function.

PROOF: By geometric series, we have

$$\begin{aligned} \frac{1}{1+\alpha t} &= \frac{1}{1-(-\alpha t)} = 1 + (-\alpha t) + (-\alpha t)^2 + (-\alpha t)^3 + \ldots \\ &= 1 - \alpha t + \alpha^2 t^2 - \alpha^3 t^3 + \ldots \end{aligned}$$

Since $\frac{1}{1+\alpha t} = (1+\alpha t)^{-1}$, we have that the derivative of $\frac{1}{1+\alpha t} = (1+\alpha t)^{-1}$ is $-\alpha \frac{1}{(1+\alpha t)^2}$. The derivative of the power series for this function $\frac{1}{1+\alpha t}$ is

$$\begin{aligned} \frac{d}{dt}(1 + (-\alpha t) + (-\alpha t)^2 + (-\alpha t)^3 + \ldots) &= -\alpha + 2\alpha^2 t - 3\alpha^3 t^2 + \ldots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)\alpha^{n+1} t^n. \end{aligned}$$

(Fortsettes på side 10.)

Hence

$$-\alpha\frac{1}{(1+\alpha t)^2} = \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\alpha^{n+1}t^n.$$

Therefore, $-\alpha a_n = (-1)^{n+1}(n+1)\alpha^{n+1}$ for all n.

If $\alpha = 0$, then one can check that $a_n = 0$ for all n. In this case $a_n = O(n)$.

If $\alpha \neq 0$, then we can divide it both sides and find that $a_n = (-1)^n(n+1)\alpha^n$ for all n. Since $n+1 = O(n)$ and $(-1)^n\alpha^n = O(1)$ for $0 \leq \alpha \leq 1$, we get $a_n = O(n)O(1) = O(n)$.

b (vekt 10%)

Let $\{x_n\}_{n=0,1,2,\ldots}$ be a sequence satisfying the following conditions: $x_0 = 1$, $x_1 = 3$, $x_2 = 2$, and for all $n \geq 3$ we have:

$$x_n = -3x_{n-1} + 4x_{n-3}.$$

Let $F(t)$ be the generating function of the sequence $\{x_n\}$, that is

$$F(t) = \sum_{n=0}^{\infty} x_n t^n.$$

Show that

$$F(t) = \frac{1+6t+11t^2}{1+3t-4t^3}.$$

Hint: Find a functional equation satisfied by $F(t)$, and use that to solve for $F(t)$.

PROOF: We write explicitly the generating function

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} x_n t^n = 1 + 3t + 2t^2 + (-3x_2 + 4x_0)t^3 + (-3x_3 + 4x_1)t^4 + (-3x_4 + 4x_2)t^5 + \ldots \\ &= 1 + 3t + 2t^2 + (-3x_2t^3 - 3x_3t^4 - 3x_4t^5 + \ldots) \\ &\quad +(4x_0t^3 + 4x_1t^4 + 4x_2t^5 + \ldots) \end{aligned}$$

We calculate the summands in the RHS of the above equality.

$$\begin{aligned} (-3x_2t^3 - 3x_3t^4 - 3x_4t^5 + \ldots) &= -3t(x_2t^2 + x_3t^3 + \ldots) = -3t(F(t) - x_0 - x_1t) \\ &= -3tF(t) + 3x_0t + 3x_1t^2 = -3tF(t) + 3t + 9t^2. \end{aligned}$$

$$(4x_0t^3 + 4x_1t^4 + 4x_2t^5 + \ldots) = 4t^3(x_0 + x_1t + \ldots) = 4t^3F(t).$$

Hence

$$F(t) = (1 + 3t + 2t^2) + (-3tF(t) + 3t + 9t^2) + 4t^3F(t).$$

Moving the terms with $F(t)$ to the LHS, we find $F(t)(1 + 3t - 4t^3) = 1 + 6t + 11t^2$. Therefore,

$$F(t) = \frac{1+6t+11t^2}{1+3t-4t^3}.$$

(Fortsettes på side 11.)

c (vekt 10%)

Show that the partial fraction decomposition for the above $F(t)$ is:

$$F(t) = \frac{2}{1-t} - \frac{3}{2(1+2t)} + \frac{1}{2(1+2t)^2}.$$

Then provide a closed form formula for x_{100}.

PROOF: We can factorise the denominator in the rational function representing $F(t)$ as follows: $1+3t-4t^3 = (1-t)(1+2t)^2$. Hence, there are constants a, b, c for which:

$$F(t) = \frac{a}{1-t} + \frac{b}{(1+2t)} + \frac{c}{(1+2t)^2}.$$

We now find a, b, c. We make the terms in the RHS to have the same denominator $1+3t-4t^3 = (1-t)(1+2t)^2$:

$$\begin{aligned} \frac{a}{1-t} + \frac{b}{(1+2t)} + \frac{c}{(1+2t)^2} &= \frac{a(1+2t)^2 + b(1+2t)(1-t) + c(1-t)}{1+3t-4t^3} \\ &= F(t) = \frac{1+6t+11t^2}{1+3t-4t^3}. \end{aligned}$$

Hence $a(1+2t)^2 + b(1+2t)(1-t) + c(1-t) = 1+6t+11t^2$.

Substitute $t=1$ into the above, we obtain: $9a = 18 \Rightarrow t = 18/9 = 2$.

Similarly, substitute $t=-0.5$ into the above, we obtain: $1.5c = 0.75 \Rightarrow c = 0.75/1.5 = 0.5$.

Taking the derivative of both sides of the above by t, and then substitute $t=-0.5$, we find by using the values $a=2$ and $c=0.5$, that $b=-1.5$.

We have:

$$\begin{aligned} \frac{2}{1-t} &= 2(1+t+t^2+\ldots) = 2\sum_{n=0}^{\infty} t^n, \\ -\frac{3}{2(1+2t)} &= -\frac{3}{2(1-(-2t))} = -\frac{3}{2}(1-2t+4t^2-8t^3+\ldots) \\ &= -\frac{3}{2}\sum_{n=0}^{\infty}(-2)^n t^n, \\ \frac{1}{2(1+2t)^2} &= \frac{-1}{4}\frac{d}{dt}\left(\frac{1}{(1+2t)}\right) = -\frac{1}{4}\sum_{n=0}^{\infty}(-2)^{n+1}(n+1)t^n \end{aligned}$$

Since $F(t)$, which is the generating function of the sequence x_n, is the sum of the above 3 expressions, we obtain:

$$x_n = 2 - \frac{3}{2}(-2)^n - \frac{1}{4}(n+1)(-2)^{n+1}.$$

x_{100} is obtained from the above by substituting $t=100$.

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