MAT3250 - Introduction (Sec. 1.1)

Me: Jørgen

Lectures Mon & Tues, "problem sessions" Wed (Anne Bruvgård)

1 mandatory assignment (13 Mar), final exam (11 June)

Student representative: One of you

Discrete Mathematics = "mathematics of finite sets"

This course in 3 parts

① Counting

② Graphs 

③ "Algebraic systems", e.g. finite sets S with addition & multiplication (see MAT2200)

Counting: Let S be a finite set, e.g. $S=\{1,2,5\}$ $(|S|=3)$

A counting problem asks for $|S|$ = # of elements of S
= cardinality of S

Often have sets $S_1, S_2, S_3, \ldots$,
want understanding of $|S_1|, |S_2|, |S_3|, \ldots$

- formula for $|S_k|$?
- formula for $\sum_k |S_k| z^k$?
- Relation between $|S_k|$?
- Growth of $|S_k|$ as $k \to \infty$?

Example Let R be a set with $|R|=n$, let S be the set of k-element subsets of R $(0 \le k \le n)$, what is $|S|$?

E.g. $R=\{1,2,3,4\}$, $k=2$, $S=\{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ $|S|=6$

In general, $|S|=\binom{n}{k}=\frac{n!}{k!(n-k)!}$, binomial coefficient

Example Let $n \geq 1$. In how many distinct* ways can n be written as a sum of positive integers?

* reordered sums are not considered distinct

n	Sums	$p(n)$ # of sums
1	1	1
2	1+1, 2	2
3	3, 1+1+1, 1+2	3
4	4, 3+1, 2+2, 2+1+1, 1+1+1+1	5

$$1 + \sum_{n=1}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

Basic counting tools

Products: Let A, B be sets. Then $A \times B$ is set of pairs (a,b) $a \in A$ $b \in B$

For sets $A_1, \ldots, A_n$, $A_1 \times \cdots \times A_n$ is set of n-tuples $(a_1, \ldots, a_n)$ $a_i \in A_i$

Rule of product: $|A \times B| = |A||B|$, $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$.

Example A restaurant serves 3 starters, 4 mains & 2 desserts.

How many 3-course meals?

Interpret "meal" as triple (starter, main, dessert), so

$$|\{\text{meals}\}| = |(\text{starters, main, dessert})| \overset{\text{R.O.P}}{=} |\{\text{starter}\}|\,|\{\text{main}\}|\,|\{\text{dessert}\}|$$
$$= 3 \times 4 \times 2 = 24.$$

Sums Sets A & B are disjoint if $A \cap B = \emptyset$

Rule of summation If A & B are disjoint, then $|A \cup B| = |A| + |B|$

If $A_1, \ldots, A_n$ are such that $\forall i \neq j$, A_i & A_j are disjoint, then $|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|$.

Example How many pairs of integers (m,n) exist with

- $1 \le m, n \le 10$
- At least one of m, n is odd

Let $A = \{(m,n),\ 1 \le m,n \le 10\}$ $\quad |A| = 100$

$B = \{(m,n),\ 1 \le m,n \le 10,$ at least one is odd$\}$

$C = A \setminus B = \{(m,n),\ 1 \le m,n \le 10,$ both are even$\}$

5 even integers in $[1,10] \Rightarrow |C| = 5 \times 5 = 25$

Rule of sum $\Rightarrow |A| = |B| + |C| \Rightarrow |B| = |A| - |C| = 100 - 25 = 75.$

Rule of equality Let A, B be sets, & let $f: A \to B$ be a function. Then f is a bijection if for every $b \in B$ there is exactly one $a \in A$ s.t. $f(a) = b$.

Rule of equality: If A, B are finite sets & $f: A \to B$ a bijection, then $|A| = |B|$

Proposition A function $f: A \to B$ is a bijection if and only if there is a function $g: B \to A$ such that

$$f(g(b)) = b \qquad \forall b \in B$$
$$g(f(a)) = a \qquad \forall a \in A.$$

Example Let $n \ge 1$ have prime factorisation $n = p_1^{e_1} \cdots p_r^{e_r}$

How many d with $1 \le d \le n$ divide n? (e.g. $12 = 2^2 3^1$)

Every such d is of the form $d = p_1^{f_1} \cdots p_n^{f_n} \quad 0 \le f_i \le e_i \ \forall i.$

Get bijection

$$A = \{(f_1, \dots, f_r) \mid 0 \le f_i \le e_i\} \longleftrightarrow \{1 \le d \le n \mid d \text{ divides } n\}$$
$$(f_1, \dots, f_n) \longmapsto p_1^{f_1} \cdots p_r^{f_r}$$

Answer is

$$|A| = (1+e_1)(1+e_2)\cdots(1+e_r).$$

Eg. $n=12$ has divisors $\{1,2,3,4,6,12\}$, $6=(1+2)(1+1)$

Double counting Let A, B be sets. An **incidence relation** between A & B is a subset $I \subseteq A \times B$.
If $(a,b) \in I$, say a & b are **incident**, & write aIb.

Rule of double counting With A, B, I as above,

$$\text{define} \begin{cases} \text{for } a \in A, & r(a) = \#\, b \in B \text{ s.t. } aIb \\ \text{for } b \in B & r(b) = \#\, a \in A \text{ s.t. } aIb \end{cases}$$

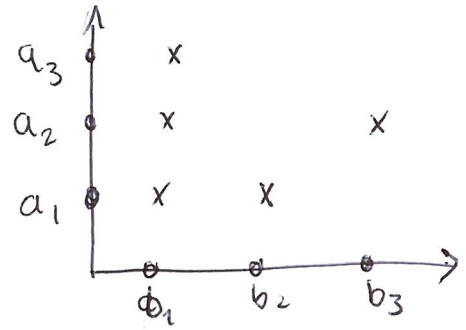
Then $\sum_{a\in A} r(a) = \sum_{b\in B} r(b)$.

Proof $I \subseteq A \times B$

$$\bigcup_{b\in B} \{(a,b) \mid aIb\} = I = \bigcup_{a\in A} \{(a,b) \mid aIb\}$$

$$\sum r(b) = \sum_{b\in B} |\text{—}''\text{—}| = |I| = \sum_{a\in A} |\text{—}''\text{—}| = \sum_{a\in A} r(a)$$

$5 = 1+2+2 = 3+1+1$ $\rightsquigarrow$



Example Let $\varphi(n) = \#$ of divisors of n. What is $t(n) = \varphi(1) + \cdots + \varphi(n)$?
Define incidence relation $d I m$ if d divides m

d \ m	1	2	3	4	5	6
1	×	×	×	×	×	×
2		×		×		×
3			×			×
4				×		
5					×	
6						×

Let $r(d) = \#\, m$ with $1 \le m \le n$ s.t. dIm. Then $r(d) = \lfloor \frac{n}{d} \rfloor$, &
$\varphi(1) + \cdots + \varphi(n) = r(1) + \cdots + r(n)$.

$$= \lfloor \tfrac{n}{1} \rfloor + \cdots + \lfloor \tfrac{n}{n} \rfloor$$

$$\approx n\left(\tfrac{1}{1} + \cdots + \tfrac{1}{n}\right) \approx n \log n$$