MAT3250 - Sections 1.2 & 1.3

Terminology: An n-set is a set with n elements, e.g. $\{1, _, n\}$.

Example $\binom{n}{k}$ is the # of k-subsets of an n-set.

A set partition of a set A is a division of elements of A into subsets, e.g.: $A = \{\{1,2,3\}\}$. The set partitions of A are

$$\{\{1,2,3\}\},\ \{\{1\},\{2\},\{3\}\},\ \{\{1,2\},\{3\}\},\ \{\{1\},\{2,3\}\},\ \{\{1,3\},\{2\}\}$$

The Stirling number $S_{n,k}$ of the 2nd kind is the number of set partitions of an n-set into k subsets

E.g. $S_{n,1} = 1$ $\quad \{\{1, .., n\}\}$

$S_{n,n} = 1$ $\quad \{\{1\}, .. \{n\}\}$

$S_{3,2} = 3$ $\quad \left(\& \ S_{n,n-1} = \binom{n}{2}\right)$

Integer partitions Let $n \geq 1$. A k-partition of n is a sum

$$n = n_1 + \cdots + n_k \qquad n_i \geq 1.$$

The number of such is $P_{n,k}$

E.g. $P_{n,n} = 1$ $\quad 1 + \cdots + 1$

$P_{n,1} = 1$ $\quad n$

$P_{4,2} = 2$ $\quad 3+1,\ 2+2$

Let N be an n-set, e.g. $N = \{1, _, n\}$. A k-permutation of N is a sequence $a_1 \ldots a_k$ where $a_i \in N$ & all a_i are distinct.

E.g. $(1,2,3)$, $(1,3,5)$, $(1,5,3)$ are 3-permutations of $\{1, _, 5\}$. How many k-permutations does an n-set have?

Want $(a_1, \ldots, a_k)$, all a_i distinct.

n choices for a_1

$(n-1)$ — " — a_2 $\quad \Rightarrow n(n-1)\ldots(n-k+1)$ k-permutations of an n-set

$n-k+1$ — " — a_k

Falling factorial: $n^{\underline{k}} \overset{\text{def}}{=} n(n-1)\cdots(n-k+1)$

Rising factorial: $n^{\overline{k}} = n(n+1)\cdots(n+k-1)$

E.g. $n! = n^{\underline{n}} = 1^{\overline{n}}$.

Ex: Binomial coefficients: For every k-subset of an n-set, $\{a_1, \ldots a_k\}$ there are $k!$ k-permutations using $a_1, \ldots a_k$, (e.g. $k=3$: $a_1a_2a_3$, $a_2a_1a_3$, $a_1a_3a_2$, $a_2a_3a_1$, $a_3a_1a_2$, $a_3a_2a_1$)

so # k subsets $= \dfrac{\#\,k\text{-permutations}}{k!}$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

Example An ordered k-partition of n is a choice of $a_1, \ldots a_k \geq 1$ such that $a_1 + \cdots + a_k = n$. Ordered 3-partitions of 4 are $1+1+2$, $1+2+1$, $2+1+1$

How many? $\binom{n-1}{k-1}$

Proof: Put n dots in a line $\bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet$

Represent ordered partition by placing + signs in gaps between dots $\quad 2+1+3 \longleftrightarrow \bullet\ \bullet + \bullet + \bullet\ \bullet\ \bullet$

$n-1$ gaps, $k-1$ "+" signs $\Longrightarrow \binom{n-1}{k-1}$ ways of doing this

A formula with $S_{n,k}$: Let $r, k, n \geq 1$. Then

$$r^n = \sum_{k=1}^{n} r^{\underline{k}} S_{n,k}$$

Let $|N| = n$, $|R| = r$. A function $f: N \to R$ has r choices for the value of each elt. of $N \Longrightarrow \underline{r^n}$ functions.

Divide set of such f by $f(N) \subseteq R$.

There are $\binom{r}{k}$ possibilities for $f(N)$ with $|f(N)| = k$.

Given $K \subseteq R$, how many f have $f(N) = K$?

Matches ordered k-set partitions of N.

"Ordered k-set partition of N" is sequence $A_1, A_2, \ldots, A_k$ such that the A_i are disjoint & $A_1 \cup \cdots \cup A_n$

Ordering matters, so $\{\{1,2\},\{3\}\}$ & $\{\{3\},\{1,2\}\}$ are different ordered 2-set partitions of $\{1,2,3\}$.

For each k-set partition of N, there are $k!$ ways of making it ordered,

so # ordered k-set partitions of $N = k! \times$ #(unordered k-set partitions of N)

$$= k!\, S_{n,k}$$

Let $K = \{b_1, \dots, b_k\} \subseteq R$.

If $f: N \to R$ has $f(N) = K$, define ordered k-set partition of N by

$A_1 = f^{-1}(b_1) = \{a \in N \mid f(a) = b_1\} \subseteq N$

$\vdots$

$A_k = f^{-1}(b_k)$.

$\{A_1, \dots, A_k\}$

Given ordered k-set partition of N, define $f: N \to R$ by

$f(a) = b_i$ if $a \in A_i$

Get bijection

$$\{\text{Ordered } k\text{-set part. of } N\} \longleftrightarrow \left\{ f: N \to R \text{ with } \begin{array}{l} \\ f(N) = K \end{array} \right\},$$

so # $f: N \to R$ with $f(N) = K$ is $k!\, S_{n,k}$.

So a $f: N \to R$ is given by a choice of $K \subseteq R$ s.t. $f(N) = K$ $\left(\binom{r}{k}\right)$

& an $f: N \to R$ s.t. $f(N) = K$ $(k!\, S_{n,k})$.

Thus $$r^n = \sum_{k=1}^{n} \binom{r}{k} k!\, S_{n,k} = \sum_{k=1}^{n} \frac{r^{\underline{k}}}{k!} k!\, S_{n,k} = \sum_{k=1}^{n} r^{\underline{k}} S_{n,k}.$$

Permutations Let N be an n-set. A permutation of N is a bijection $\pi: N \to N$. Assume $N = \{1, \dots, n\}$, then a permutation can be presented in various ways:

2-line matrix: $\begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & & \pi(n) \end{pmatrix}$

Bijectivity means every element of first row appears exactly once on 2nd row

Permutation Can represent π as a word of length n

$$\pi(1)\pi(2)\cdots\pi(n),$$

where each of $1, 2, \ldots, n$ appears once.

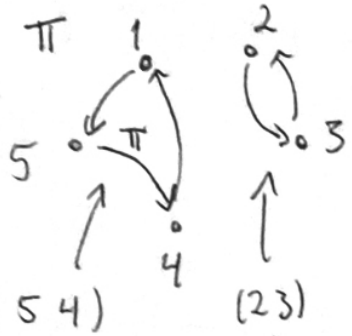
E.g. permutations of $\{1,2,3\}$ are 123, 213, 132, 231, 312, 321.

$\rightsquigarrow$ n choices for $\pi(1)$, $(n-1)$ for $\pi(2)$, ... $\rightsquigarrow$ $n!$ permutations.

Cycles: A k-cycle of a permutation π is a sequence $(a_1\ a_2\ \ldots\ a_k)$ such that $\pi(a_1)=a_2$, $\pi(a_2)=a_3$, ... $\pi(a_{k-1})=a_k$, $\pi(a_k)=a_1$

A permutation can be described by listing its cycles.

Examples

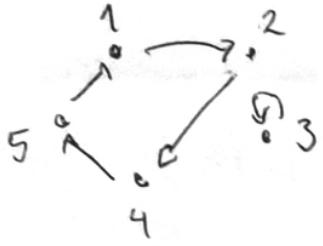


(1 5 4) cycle (23) cycle

π is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$ (2-line matrix)

or 5 3 2 1 4 (word)

or (1 5 4)(2 3) (cycles)



$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$

2 4 3 5 1

(1 2 4 5)(3)

Note Cycle representation is not unique $(1\ 5\ 4)(2\ 3) = (1\ 5\ 4)(3\ 2)$
$= (2\ 3)(5\ 4\ 1)$ etc.

Def Sterling numbers of the first kind $S_{n,k}$ are the number of permutations of an n-set with exactly k cycles

E.g. $S_{n,n}=1$ $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$, $= (1)(2)\cdots(n)$

$S_{n,n-1} = \binom{n}{2}$, every permutation with $n-1$ cycles looks like

$\pi(i)=j$ $\pi(j)=i$ $\pi(k)=k$ for all $k \neq i$ or j

$\binom{n}{2}$ choices of i, j.

$S_{n,1} = (n-1)!$ $\pi(1) \neq 1$, since would then have ≥ 2 cycles $\rightsquigarrow$ $n-1$ choices for $\pi(1)$

$\pi(\pi(1)) \neq 1, \pi(1)$ $\rightsquigarrow$ $n-2$ choices for $\pi(\pi(1))$

$\vdots$

$(n-1) \times (n-2) \times \cdots \times 1$ choices for permutation.

$S_{n,n}$



$S_{n,n-1}$



$S_{n,1}$

