Recurrence relations & "polynomial method" Sec. 1.4

Given a sequence of numbers, a <u>recurrence relation</u> relates each number to previous numbers in the sequence.

<u>Example</u> For $n, k \geq 1$, have $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Can show using formula $\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$, also <u>counting proof</u>:

$\binom{n}{k}$ = # k-subsets S of $\{1, \ldots, n\}$.

Two kinds of S: Either (1) $n \in S$ or (2) $n \notin S$.

In case (1), $S = \{n\} \cup S'$, where S' is $(k-1)$-subset of $\{1, \ldots, n-1\}$

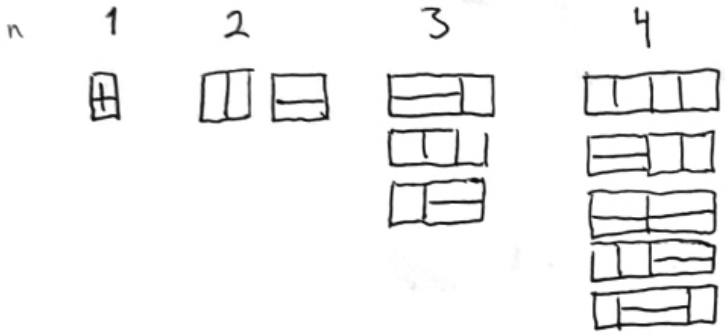
In case (2), S is a k-subset of $\{1, \ldots, n-1\}$

Case (1) $\rightsquigarrow \binom{n-1}{k-1}$ subsets Case (2) $\rightsquigarrow \binom{n-1}{k}$ subsets

$\Rightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$



<u>Example</u> Let C_n be the # of ways to cover a $(2 \times n)$ rectangle with (1×2) rectangles

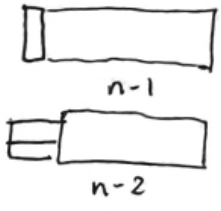


C_n 1 2 3 5

<u>Recursion</u>: The left hand side of a covering is either (1) ▯ or (2) ▭

Ways of covering the remainder is C_{n-1} in case (1)

C_{n-2} in case (2)



Thus $C_n = C_{n-1} + C_{n-2}$ (Fibonacci!)

Stirling nos of 2nd kind $S_{n,k}$ = # partitions of n-set into k subsets.

Recurrence relation: Partition $\{1, \dots, n\}$ into k subsets

2 cases: I n sits alone, i.e. $\{n\}$ is in the partition (Ex: $\{\{1,2\},\{3\},\{4\}\}$)

II $\{n\}$ is not in the partition (Ex: $\{\{1\},\{2,4\},\{3\}\}$)

In case I, partition determined by a $(k-1)$-partition of $\{1, \dots n-1\}$

$(\{\{1,2\},\{3\}\} \longleftrightarrow \{\{1,2\},\{3\},\{4\}\})$, $S_{n-1,k-1}$ such

In case II, "forgetting" n gives a k-partition of $\{1, \dots, n-1\}$.

$S_{n-1,k}$ such, k choices for which set n "joins" $\Rightarrow$ $k S_{n-1,k}$ such

$\{\{1\},\{2\},\{3\}\} \rightrightarrows \{\{1,4\},\{2\},\{3\}\}$, $\{\{1\},\{2,4\},\{3\}\}$, $\{\{1\},\{2\},\{3,4\}\}$ (k)

In total $S_{n,k} = \underbrace{S_{n-1,k-1}}_{\text{I}} + \underbrace{k S_{n-1,k}}_{\text{II}}$



Stirling nos of 1st kind $s_{n,k}$ = permutations of $\{1, \dots, n\}$ with k cycles

Recurrence relation: For a permutation $(a_1 \dots a_{i_1})(a_{i_1+1} \dots a_{i_2}) \dots (\quad)$

get a permutation of $\{1, \dots, n-1\}$ by deleting n.

E.g. $(1\ 2\ 4)(3\ 5) \longrightarrow (1\ 2\ 4)(3)$

2 cases: n is a fix point $\overset{\text{def}}{\iff}$ (n) appears.

$\rightsquigarrow$ deleting gives a permutation with $k-1$ cycles.

$S_{n-1,k-1}$ such.

II n is not a fix point $\rightsquigarrow$ deleting gives permutation with k cycles, $S_{n-1,k}$ such permutations of $\{1, \dots, n-1\}$

n can be added in $n-1$ ways $\rightsquigarrow$ $(n-1)S_{n-1,k}$ permutations in case II



$(1\ 2\ 4)(3)$ $\Leftarrow$ $(1\ 2\ 4\ 5)(3)$, $(1\ 2\ 5\ 4)(3)$, $(1\ 5\ 2\ 4)(3)$, $(1\ 2\ 4)(3\ 5)$

In total $S_{n,k} = S_{n-1,k-1} + (n-1)S_{n-1,k}$.

			1		
		1		1	
	2		3		1
6		11		6		1
24	50		35		10		1

Binomial theorem: For $n \geq 0$, $x \in \mathbb{R}$, have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Pf by induction on n:

$n=0$: $(1+x)^0 = 1$, ok.

Assume by induction, $(1+x)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k$. Then

$$(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)\left(\sum_{k=0}^{n-1} \binom{n-1}{k} x^k\right) = \sum_{k=0}^{n} \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right) x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Applications: $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^n = (1+1)^n = 2^n$

$$\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k = (1+(-1))^n = 0.$$

Generalised binomial coefficients: Let z be real or complex, $k \in \mathbb{Z}$

Then $\binom{z}{k} = \begin{cases} \dfrac{z(z-1)\cdots(z-k+1)}{k!} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$

Let $z^{\underline{k}} = z(z-1)\cdots(z-k+1)$, so $\binom{z}{k} = \dfrac{z^{\underline{k}}}{k!}$

Recursion still holds: $\binom{z}{k} = \binom{z-1}{k} + \binom{z}{k-1}$

Proof by "polynomial trick": Let $k \geq 0$, let $F(z) = \binom{z}{k} - \binom{z-1}{k} - \binom{z-1}{k-1}$.

We want to show $F(z) = 0 \quad \forall z \in \mathbb{C}$. By definition, $F(z)$ is a polynomial of degree $\leq k$. If $F(z)$ is not constant 0, it has $\leq k$ zeros. $F(n) = 0$ for all integer n; so $F(z) = 0 \quad \forall z \in \mathbb{C}$.

* By $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Stirling no. formulas: Recall if $r, n, k \geq 0$ are integers, then

$$r^n = \sum_{k=1}^{n} S_{n,k} \, r^{\underline{k}}$$

r^n: # of maps $f : \{1, \dots, n\}$ to $\{1, \dots, r\}$

$S_{n,k}$: # of maps with $|f(\{1, \dots, n\})| = k$.

Polynomial trick $\Rightarrow \forall z \in \mathbb{C}$, $z^n = \sum_{k=1}^{n} S_{n,k} \, z^{\underline{k}}$ (*)

$S_{n,k}$ are unique nos. such that (*) hold.

$s_{n,k}$ formula: For $z \in \mathbb{C}$, $n, k \geq 1$ integers, have

$$z^{\underline{n}} = \sum_{k=1}^{n} (-1)^{n-k} s_{n,k} \, z^k$$

Proof by induction: $n = 1$: $z^{\underline{1}} = z = s_{1,1} \, z$, o.k.

Assume by induction $z^{\underline{n}} = \sum_{k=1}^{n} (-1)^{n-k} s_{n,k} z^k$. Then

$$z^{\underline{n+1}} = z^{\underline{n}} (z - n) = \left(\sum_{k=1}^{n} (-1)^{n-k} s_{n,k} z^k \right) (z - n)$$

$$= \sum_{k=1}^{n+1} \left((-1)^{n-k} s_{n,k} z^{k+1} - (-1)^{n-k} n s_{n,k} z^k \right)$$

$$= \sum_{k=1}^{n+1} (-1)^{n-k} (s_{n,k-1} + n s_{n,k}) z^k$$

Coefficient of z^k is $(-1)^{n-k+1} s_{n,k-1} - (-1)^{n-k} n s_{n,k} = (-1)^{n+1-k} (s_{n,k-1} + n s_{n,k})$

$$= (-1)^{n+1-k} s_{n+1,k}.$$