Generating functions Sec 3.1 & 3.2

Let $c_0, c_1, c_2, \ldots$ be a sequence of numbers. The generating function of the sequence is

$$A(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$

Here z is either a complex number, or a "formal variable", so we allow non-convergent series.

Ex: $c_k = \binom{n}{k} \rightsquigarrow A(z) = \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^k$

$c_k = k! \rightsquigarrow A(z) = \sum_{k=0}^{\infty} k!\, z^k$, converges only if $z=0$.

Operations on generating functions:

- Sum: Given $A(z) = \sum_{k=0}^{\infty} a_k z^k$ & $B(z) = \sum_{k=0}^{\infty} b_k z^k$, have

$$A(z) + B(z) = \sum_{k=0}^{\infty} (a_k + b_k) z^k$$

- Product: If $A(z) = \sum_{k=0}^{\infty} a_k z^k$, $B(z) = \sum_{k=0}^{\infty} b_k z^k$, then

$$A(z)B(z) = (a_0 + a_1 z + \cdots)(b_0 + b_1 z + \cdots) = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + \cdots$$

$$= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) z^k.$$

Interpretation: If $a_i = |A_i|$, $b_i = |B_i|$, then $c_k = \sum_{j=0}^{k} a_j b_{k-j}$ is # of pairs (α, β), where $\alpha \in A_j$ & $\beta \in B_{k-j}$ for some j.

Example: Consider a 6-sided die, let a_n be the number of ways to roll the number n

n	0	1	2	3	4	5	6	≥ 7
a_n	0	1	1	1	1	1	1	0

Generating function $A(z) = z + z^2 + z^3 + z^4 + z^5 + z^6$.

Now take 2 dice, let b_k = # of ways to roll these with total number of eyes $\bullet = k$

Then $b_k = a_1 a_{k-1} + a_2 a_{k-2} + \cdots + a_{k-1} a_1$, so
(1 on 1st, $k-1$ on 2nd)

$$B(z) \overset{\text{def}}{=} \sum_k b_k z^k = A(z)A(z) = (z + \cdots + z^6)^2.$$

$$= z^2 + 2z^3 + \cdots + 6z^7 + 5z^8 + \cdots + z^{12}.$$

Now take ℓ dice, with $s_1, s_2, \ldots, s_\ell$ sides. For $i = 1, \ldots, \ell$, let $a_{i,k} = \begin{cases} 1 & \text{if } 1 \le k \le s_i \\ 0 & \text{otherwise,} \end{cases}$ $A_i(z) = \sum_k a_{i,k} z^k = (z + \cdots + z^{s_i})$

Let b_k = # of ways to roll the dice with k total eyes

$$b_k = \sum_{\substack{k_1, \ldots, k_\ell \\ \sum k_i = k}} a_{1,k_1} a_{2,k_2} \cdots a_{\ell,k_\ell} \Rightarrow B(z) = A_1(z) A_2(z) \cdots A_\ell(z).$$

Note b_k is # of ways to write $k = k_1 + k_2 + \cdots + k_\ell$ $\quad 1 \le k_i \le s_i$,
so counts (certain) ordered partitions of k of length ℓ.

Let now $c_k = \begin{cases} 1 & \text{if } k \ge 1 \\ 0 & \text{otherwise,} \end{cases}$ d_k = # of ordered partitions of k of length ℓ.

Then $d_k = \sum_{\substack{k_1, \ldots, k_\ell \\ \sum k_i = k}} c_{k_1} \cdots c_{k_\ell}$. (*)

Let $C(z) = \sum c_k z^k$, $D(z) = \sum d_k z^k$, then (*) $\Rightarrow D(z) = C(z)^\ell$.

Have $C(z) = z + z^2 + \cdots = \frac{z}{1-z}$, so $D(z) = z^\ell (1-z)^{-\ell} = z^\ell \left(\sum_{k \ge 0} \binom{-\ell}{k} (-z)^k \right)$

Have $\binom{-\ell}{k} = \frac{(-\ell)^{\underline{k}}}{k!} = \frac{(-\ell)(-\ell-1)\cdots(-\ell-k+1)}{k!} = (-1)^k \frac{(\ell+k-1)^{\underline{k}}}{k!} = (-1)^k \binom{\ell+k-1}{k}$

$$\Rightarrow D(z) = z^\ell \left(\sum_{k \ge 0} \binom{\ell+k-1}{k} z^k \right) = z^\ell \left(\sum_{k \ge 0} \binom{\ell+k-1}{\ell-1} z^k \right) = \sum_{n \ge \ell} \binom{n-1}{\ell-1} z^n$$

$\Longrightarrow$ # of ordered partitions of n of length l is $\binom{n-1}{l-1}$.