"Solving" recurrences via generating functions (Sec. 3.2)

Consider Fibonacci numbers:

$F_0 = 0 \quad F_1 = 1$

$\forall n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$

Want formula for F_n.

Remark: In linear algebra, use $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$

+ diagonalisation of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ to analyse F_n. Here alternative path to same formula.

Steps: ① Translate recurrence relation into equation for generating function. E.g., with $F(z) = \sum_{n \geq 0} F_n z^n$, have

$$F(z) = z + \sum_{n \geq 2} F_n z^n = z + \sum_{n \geq 2} (F_{n-1} + F_{n-2}) z^n$$

$$= z + z\left(\sum_{n \geq 1} F_n z^n\right) + z^2\left(\sum_{n \geq 0} F_n z^n\right) = z + z F(z) + z^2 F(z)$$

② Solve resulting equation for generating function

E.g. $F(z) = z + z F(z) + z^2 F(z)$

$$\Updownarrow$$

$$F(z)(1 - z - z^2) = z$$

$$\Updownarrow$$

$$F(z) = \frac{z}{1 - z - z^2}$$

③ Use expression for generating function to determine coefficients of function.

Here $\frac{z}{1 - z - z^2} = z\left(\frac{1}{1 - (z + z^2)}\right) = z\left(1 + (z + z^2) + (z + z^2)^2 + \cdots\right)$

leads to a formula, (see exercises), we want a better one.

A rational function is a fraction $\frac{f(z)}{g(z)}$, with $f(z)$, $g(z)$ polynomials.

How to express coefficients c_n in $\sum_n c_n z^n = \frac{f(z)}{g(z)}$?

Write $\frac{f(z)}{g(z)}$ as a sum of fractions $\frac{f_i(z)}{(1-\alpha_i z)^{e_i}}$ (partial fraction decomposition)

E.g. $\frac{1}{1-z^2} = \frac{1/2}{1-z} + \frac{1/2}{1+z}$

Finding the partial fraction decomposition of $\frac{f(z)}{g(z)}$

① Assume $g(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_n z^n$.

Have $g(z) = (1-\alpha_1 z)(1-\alpha_2 z) \cdots (1-\alpha_n z)$, where $\beta_1, \ldots, \beta_n$ are roots of $g(z)$, $\alpha_i = \frac{1}{\beta_i}$

E.g. In example $\frac{z}{1-z-z^2}$, $g(z) = 1 - z - z^2$ has roots

$$\frac{1 \pm \sqrt{1+4}}{-2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

some computation

So $\beta_1 = -\frac{1}{2} + \frac{\sqrt{5}}{2}$, $\beta_2 = -\frac{1}{2} - \frac{\sqrt{5}}{2}$ $\rightsquigarrow$ $\alpha_1 = \frac{1}{\beta_1} = \frac{1}{2} + \frac{\sqrt{5}}{2}$ $\alpha_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$

Get $(1-z-z^2) = (1-\alpha_1 z)(1-\alpha_2 z)$.

② Assume all α_i distinct. Then find $a_1, \ldots, a_n$ such that

$$\frac{1}{g(z)} = \frac{a_1}{1-\alpha_1 z} + \frac{a_2}{1-\alpha_2 z} + \cdots + \frac{a_n}{1-\alpha_n z}.$$

E.g. Find a_1, a_2 such that $\frac{1}{1-z-z^2} = \frac{a_1}{1-\alpha_1 z} + \frac{a_2}{1-\alpha_2 z}$.

Multiplying with $(1-z-z^2) = (1-\alpha_1 z)(1-\alpha_2 z)$ gives

$$1 = a_1(1-\alpha_2 z) + a_2(1-\alpha_1 z)$$

$$\Updownarrow$$

$$a_1 + a_2 = 1$$

$$a_1\alpha_2 + a_2\alpha_1 = 0$$

$$\Updownarrow \text{ computation}$$

$$a_1 = \frac{\alpha_1}{\sqrt{5}}, \quad a_2 = -\frac{\alpha_2}{\sqrt{5}}$$

So $\frac{z}{1-z-z^2} = z\left(\frac{\alpha_1/\sqrt{5}}{1-\alpha_1 z} - \frac{\alpha_2/\sqrt{5}}{1-\alpha_2 z}\right)$

$= z\left(\frac{\alpha_1}{\sqrt{5}}\left(1+\alpha_1 z+\alpha_2 z^2+\cdots\right) - \frac{\alpha_2}{\sqrt{5}}\left(1+\alpha_2 z+\alpha_2 z^2+\cdots\right)\right)$

Since $\frac{z}{1-z-z^2} = F_0 z^0 + F_1 z^1 + \cdots$, get $F_n = \frac{\alpha_1^n}{\sqrt{5}} - \frac{\alpha_2^n}{\sqrt{5}}$ if $n \geq 1$.

Method: Counting problem gives sequence (a_i)

↝ Find recurrence relation (e.g. $a_n = a_{n-1} + a_{n-2}$)

↝ Determine $A(z) = \sum a_i z^i$

↝ If $A(z)$ is rational function, find partial fraction decomposition

⟶ From partial fraction decomp, get formula for a_n

(e.g. $A(z) = \sum_{i=1}^{\ell} \frac{c_i}{1-\gamma_i z} \Rightarrow a_n = \sum_{i=1}^{\ell} c_i \gamma_i^n$.

Thm (Parts of Thm. 3.1) Let $q_1, q_2, \ldots, q_d \in \mathbb{C}$, $d \geq 1$, let $q(z) = 1 + q_1 z + \cdots + q_d z^d$, & let $f: \mathbb{N} \to \mathbb{C}$ be a counting function. The following 2 conditions are equivalent.

(1) For all $n \geq 0$, we have

$$f(n+d) + q_1 f(n+d-1) + \cdots + q_d f(n) = 0$$

(2) We have

$$F(z) \overset{\text{def}}{=} \sum_{n \geq 0} f(n) z^n = \frac{p(z)}{q(z)}$$

(*) Set $f(n) = 0$ for $n < 0$

for some polynomial $p(z)$ of degree $\leq d$.

Pf Assume (1). Let $G(z) = F(z)\, q(z)$. Then

$$G(z) = (1 + q_1 z + \cdots + q_d z^d)\left(\sum_{n \geq 0} f(n) z^n\right) \overset{(*)}{=} (1 + q_1 z + \cdots + z^d)\left(\sum_{n \in \mathbb{Z}} f(n) z^n\right)$$

$$= \sum_{n \in \mathbb{Z}} \left(f(n) + q_1 f(n-1) + q_2 f(n-2) + \cdots + q_d f(n-d)\right) z^n$$

If $n \geq d$, then $f(n) + q_1 f(n-1) + \cdots + q_d f(n-d) = 0$ by assumption

If $n < 0$ then $f(n) = f(n-1) = \cdots = f(n-d) = 0$ (by definition)

So the coefficients of $G(z)$ are non-zero only for terms z^n with $0 \leq n < d \Rightarrow G(z) = p(z)$ is a polynomial of degree $< d$.

②⇒① Have $p(z) = F(z)q(z) = \sum_n (f(n) + f(n-1)q_1 + \cdots + q_d f(n-d)) z^n$

Since $\deg p(z) < n$, have $f(n) + q_1 f(n-1) + \cdots + q_d f(n-d) = 0 \quad \forall n \geq d$,

which means ① is true.

Generating functions of partitions of integers

$(Q_{0,k} \overset{\text{def}}{=} 1)$

Let $n, k \geq 0$, let $Q_{n,k} = \#$ of partitions of n with all parts $\leq k$

E.g. $Q_{n,1} = 1$, $n = \overbrace{1 + \cdots + 1}^{n}$ only partition

$Q_{1,2} = 1$ $\quad 1 = 1$

$Q_{2,2} = 2$ $\quad 2 = 2, \ 1+1$

$Q_{3,2} = 2$ $\quad 3 = 2+1, \ 1+1+1$

$Q_{4,2} = 3$ $\quad 4 = 2+2, \ 2+1+1, \ 1+1+1+1$

$$Q_k(z) = \sum_{n \geq 0} Q_{n,k} z^n, \quad \rightsquigarrow Q_1(z) = \sum_{n \geq 0} z^n = \frac{1}{1-z}$$

Claim: $Q_2(z) = \frac{1}{1-z} \frac{1}{1-z^2} = (1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)$

$(1 + z + z^2 + \cdots)(1 + z^2 + z^4 + \cdots)$, coefficient of z^n from

$$z^n \cdot 1 + z^{n-2} \cdot z^2 + z^{n-4} z^4 + \cdots + 1 \cdot z^n$$

$1+1+\cdots+1$ $\quad 1+\cdots+1+2$ $\quad 1+\cdots+1+2+2$ $\quad \underbrace{2+\cdots+2}_{n/2}$

$\rightsquigarrow$ coeff. of z^n is $Q_{n,2}$.

Similarly $Q_k(z) = \frac{1}{1-z} \frac{1}{1-z^2} \cdots \frac{1}{1-z^k} = \left(\sum_{n \geq 0} z^n\right)\left(\sum_{n \geq 0} z^{2n}\right) \cdots \left(\sum_{n \geq 0} z^{kn}\right)$

Degree n part of $\frac{1}{1-z}\cdots\frac{1}{1-z^k}$ is sum of terms $z^{a_1}z^{2a_2}z^{3a_3}\cdots z^{ka_k}$

with $\sum_{i=1}^{k} i a_i = n$. $(a_1, a_2, \ldots, a_k) \longleftrightarrow \underbrace{1+1+\cdots+1}_{a_1}+\underbrace{2+\cdots+2}_{a_2}+\cdots+\underbrace{k+\cdots+k}_{a_k}$

gives $Q_{n,k}$ as coefficient.

Recall P_n = # of partitions of n

Euler (1748): $\sum_{n\geq 0} P_n z^n = \prod_{i>0} \frac{1}{(1-z^i)}$.

Remark: Formula for $Q_k(z)$ gives recurrence formula for $Q_{n,k}$

E.g. $Q_2(z) = \frac{1}{(1-z)(1-z^2)} = \frac{1}{1-z-z^2+z^3}$

$\Rightarrow Q_{n,2} - Q_{n-1,2} - Q_{n-2,2} + Q_{n-3,2} = 0 \quad \forall n \geq 3$

Euler's formula gives

Exponential generating functions (Sec. 3.3)

Let $a_0, a_1, \ldots$ be a counting function. The exponential generating function is $\hat{A}(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$.

Again operations on sequence $(a_n) \longleftrightarrow$ operations on $\hat{A}(z)$

(a_i), let $b_i = a_{i+1} \rightsquigarrow \hat{B}(z) = \sum_{n\geq 0} \frac{b_n}{n!} z^n = \sum_{n\geq 0} \frac{a_{n+1}}{n!} z^n$

$= \sum_{n\geq 0} (n+1) \frac{a_{n+1}}{(n+1)!} z^n = \hat{A}'(z)$.

$(a_i), (b_i)$ $c_i = a_i + b_i \rightsquigarrow \hat{C}(z) = \hat{A}(z) + \hat{B}(z)$

$(a_i), (b_i)$, $\hat{C}(z) = \hat{A}(z)\hat{B}(z) = \sum_{n\geq 0} \frac{c_n}{n!} z^n$, what is c_n?

Then

$$\hat{A}(z)\hat{B}(z) = \left(\sum_n \frac{a_n}{n!} z^n\right)\left(\sum_n \frac{b_n}{n!} z^n\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} \frac{a_i}{i!}\frac{b_{n-i}}{(n-i)!}\right) z^n$$

$$\Rightarrow c_n = n!\left(\sum_{i=0}^{n} \frac{a_i b_{n-i}}{i!\,(n-i)!}\right), \quad \frac{n!}{i!\,(n-i)!} = \binom{n}{i} \Rightarrow c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

Operation If $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, say (c_n) is binomial convolution of (a_n) & (b_n).

Example: Let $a_n = a^n$, $b_n = b^n$. Then $\hat{A}(z) = \sum_{n\geq 0} \frac{a^n}{n!} z^n = \sum_{n\geq 0} \frac{(az)^n}{n!} = e^{az}$

$$\hat{B}(z) = e^{bz}.$$

Binomial convolution $c_n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ (1)

$$\hat{C}(z) = \hat{A}(z)\hat{B}(z) = e^{az}e^{bz} = e^{(a+b)z} = \sum \frac{(a+b)^n}{n!} z^n.$$

$$\Rightarrow c_n = (a+b)^n \quad (2)$$

(1) = (2), new proof of binomial theorem.

Example (Derangements) A permutation π of $\{1, 2, \ldots, n\}$ is a derangement if it has no fix points (i.e. k with $\pi(k) = k$)

Let D_n = # of derangements

n	Derangements	D_n
1	None	0
2	(1 2)	1
3	(1 2 3), (1 3 2)	2
4	(1 2 3 4) (1 2 4 3) (1 3 2 4) (1 3 4 2) (1 4 2 3) (1 4 3 2) (1 2)(3 4) (1 3)(2 4) (1 4)(2 3)	9

Recurrence relation for D_n

- Partition set of derangements by length of cycle containing 1

How many permutations π of $\{1, 2, \ldots, n\}$ with k fix points?

Count as $\binom{n}{k}$ ways to choose fix points $\times$ D_{n-k} derangements of non-fix points

Since we have $n!$ partitions in total, get

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

This means sequence $(n!)$ is binomial convolution of sequences (1) & (D_n). Get

$$\sum_{n=0}^{\infty} \frac{n!}{n!} z^n = \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n \right) \left(\sum_{n=0}^{\infty} \frac{D_n}{n!} z^n \right)$$

$$\frac{1}{1-z} = e^z \hat{D}(z)$$

$$\hat{D}(z) = \frac{e^{-z}}{1-z} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n \right) \left(\sum_{n=0}^{\infty} z^n \right)$$

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$D_n = \sum_{k=0}^{n} (-1)^k \underline{n^{(n-k)}}$$