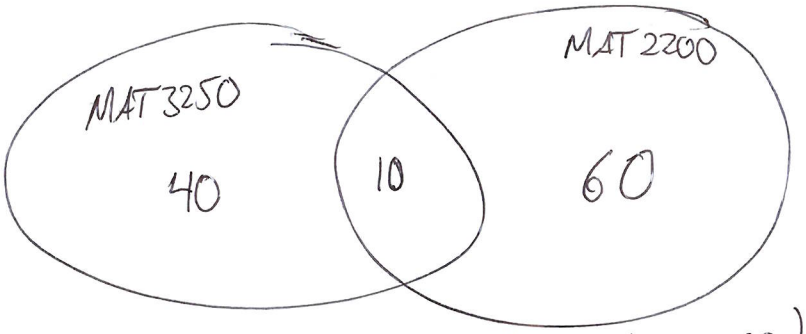
Suppose 40 students are signed up for MAT 3250, 60 students are signed up for MAT 2200, & 10 students are signed up for both. How many students take at least one of the courses?



$60+40$ is the number of pairs (student, course), but this counts students who take both courses twice. Correcting for this gives $40+60-10=90$.

Thm (Inclusion-exclusion principle for 2 sets) Let $E_1, E_2 \subseteq S$ be finite sets. Then

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|. \quad (*)$$

Pf Informal (but correct): $|E_1| + |E_2|$ is the number of pairs (x, E_i) with $i=1$ or 2 & $x \in E_i$. This counts every element in $E_1 \cup E_2$, but counts elements $x \in E_1 \cap E_2$ twice. Correcting for this gives $(*)$.

Formal: $E_1 \cup E_2 = E_1 \setminus E_2 \cup E_2 \setminus E_1 \cup E_1 \cap E_2$, where the union is disjoint

$$E_1 = E_1 \setminus E_2 \cup E_1 \cap E_2$$

$$E_2 = E_2 \setminus E_1 \cup E_1 \cap E_2$$

$$|E_1 \cup E_2| = |E_1 \setminus E_2| + |E_2 \setminus E_1| + |E_1 \cap E_2| = \big(|E_1| - |E_1 \cap E_2|\big) + \big(|E_2| - |E_1 \cap E_2|\big) + |E_1 \cap E_2|$$

$$= |E_1| + |E_2| - |E_1 \cap E_2|.$$

Example: How many numbers n with $1 \le n \le 99$ have contain exactly one 1 in decimal representation?

$S = \{1, 2, \ldots, 99\}$

$E_1 = \{n \in S \mid n$ contains 1 in 10's digit$\} = \{10, 11, \ldots, 19\}$

$E_2 = \{n \in S \mid n$ — " — 1's digit$\} = \{1, 11, 21, \ldots, 91\}$

$E_1 \cap E_2 = \{11\}$

$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2| = 10 + 10 - 1 = 19.$

Consider $E_1, E_2, \ldots, E_n \subseteq S$

Thm (Inclusion-exclusion principle)

$$(*)\quad |E_1 \cup E_2 \cup \cdots \cup E_n| = |E_1| + |E_2| + \cdots + |E_n| - |E_1 \cap E_2| - |E_1 \cap E_3| - \cdots - |E_{n-1} \cap E_n| + \cdots$$

$$= \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}| \right)$$

Ex $(n=3)$ $|E_1 \cup E_2 \cup E_3| = |E_1| + |E_2| + |E_3| - |E_1 \cap E_2| - |E_1 \cap E_3| - |E_2 \cap E_3| + |E_1 \cap E_2 \cap E_3|$

Pf

From inclusion exclusion principle for $n=2$ get

$$|E_1 \cup \cdots \cup E_{n-1} \cup E_n| = |E_1 \cup \cdots \cup E_{n-1}| + |E_n| - |(E_1 \cup \cdots \cup E_{n-1}) \cap E_n|$$

$$= |E_1 \cup \cdots \cup E_{n-1}| + |(E_1 \cap E_n) \cup (E_2 \cup E_n) \cup \cdots \cup (E_{n-1} \cap E_n)| + |E_n|$$

By induction, can write first 2 terms as a sum using I-E principle. This gives $(*)$

Example How many numbers n with $1 \le n \le 30$ are divisible by 2, 3 or 5?

$E_1 = \{n \text{ divisible by } 2\}$

$E_2 = \{ - " - \quad 3\}$

$E_3 = \{ - " - \quad 5\}$

Then $E_1 \cap E_2 = \{n \text{ divisible by } 2 \& 3\} = \{n \text{ divisible by } 6\}$

$E_2 \cap E_3 = \{n \text{ divisible by } 15\}$

$E_1 \cap E_3 = \{n \text{ divisible by } 10\}$

$E_1 \cap E_2 \cap E_3 = \{n \text{ div. by } 2 \cdot 3 \cdot 5 = 30\}$

Get $30/2 + 30/3 + 30/5 - 30/6 - 30/15 - 30/10 + 30/30 = 8$

$$|E_1 \cup E_2 \cup E_3| = |E_1| + |E_2| + |E_3| - |E_1 \cap E_2| - |E_2 \cap E_3| - |E_1 \cap E_3| + |E_1 \cap E_2 \cap E_3|$$

$(\{1, 7, 11, 13, 17, 19, 23, 29\})$

Example Recall a permutation π of $\{1, 2, \dots, n\}$ is a derangement if it has no fix points.

D_n = # of derangements.

Compute via inclusion-exclusion principle

Let $S = \{\pi$ a permutation of $\{1, \dots, n\}\}$, for $i = 1, 2, \dots, n$.

let $E_i \subseteq S$ be the set of π such that $\pi(i) = i$.

Then $S \setminus \left(\bigcup_{i=1}^n E_i\right)$ is the set of derangements, so

$$D_n = \left|S \setminus \left(\bigcup_{i=1}^n E_i\right)\right| = |S| - \left|\bigcup_{i=1}^n E_i\right| = n! - \left|\bigcup_{i=1}^n E_i\right|.$$

If $1 \le i_1 < i_2 < \dots < i_k \le n$, $E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}$ is the set of permutations π such that

$\pi(i_1) = i_1$, $\pi(i_2) = i_2$, $\dots$, $\pi(i_k) = i_k$.

The remaining $n-k$ elements can be permuted in any way, so

$$|E_{i_1} \cap \cdots \cap E_{i_k}| = (n-k)!$$

The inclusion-exclusion principle now gives

$$|E_1 \cup \cdots \cup E_n| = \sum_{k=1}^{n} (-1)^{k-1} \left(\sum_{1 \le i_1 < \cdots < i_k \le n} |E_{i_1} \cap \cdots \cap E_{i_k}| \right)$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} (n-k)!$$

There are $\binom{n}{k}$ choices of $1 \le i_1 < \cdots < i_k \le n$, so get

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!,$$

&

$$D_n = n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$$