ASYMPTOTIC ANALYSIS (SECTION 5.3)

Given a function $f\colon \mathbb{N} \to \mathbb{R}$, arising from a counting problem or otherwise, we are interested in the question of "roughly how fast does it grow". Usually this means "how quickly does it grow compared to other functions?" We have various ways of formalising this.

Definition. Let $f, g\colon \mathbb{N} \to \mathbb{R}$. We say that g grows faster than f, and write $f \prec g$, if, for every $\epsilon > 0$, there is some N such that for all $n \geq N$, we have $|f(n)| \leq \epsilon |g(n)|$.

Remark. In the common case that $g(n)$ is not equal to 0 sufficiently big n, the condition $f(n) \prec g(n)$ is equivalent to

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$$

Example. We have $n \prec n^2$, since $\lim_{n\to\infty} n/n^2 = \lim_{n\to\infty} 1/n = 0$.

Example. We have $\log n \prec n$, since

$$\lim_{n\to\infty} \frac{\log n}{n} \overset{L'Hôpital}{=} \lim_{n\to\infty} \frac{1/n}{1} = 0$$

Example. If $c, d \geq 1$, we have

$$d^n \prec n^c,$$

where $n^c/d^n \to 0$ as $n \to \infty$ can be seen from taking logarithms, since

$$\log(n^c/d^n) = c \log n - n \log d \to -\infty$$

as $n \to \infty$.

Remark. One can check that the relation $\prec$ is "transitive", that is if $f(n) \prec g(n)$ and $g(n) \prec h(n)$, then $f(n) \prec h(n)$.

0.1. **Big O notation.**

Definition. Let $g\colon \mathbb{N} \to \mathbb{R}$. Then we define

$$O(g(n)) = \{f(n)\colon \mathbb{N} \to \mathbb{R} \mid \text{ there exist } n_0, C \text{ such that } |f(n)| \leq C|g(n)| \text{ if } n \geq n_0\}$$

Informally, the set $O(g(n))$ is the set of functions which, up to multiplying by some constant, grow at most as fast as $g(n)$.

Remark. Even though $O(g(n))$ is a set, people often write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$. One reads this as "$f(n)$ is of order $g(n)$".

Remark. If $g(n)$ is never equal to 0, then $f(n) \in O(g(n))$ if and only if there is some constant C such that $|f(n)|/|g(n)| \leq C$ for all n.

Example. If $f(n) \prec g(n)$, then $f(n) \in O(g(n))$.

Example. The notation $O(g(n))$ is often used in describing how the complexity or running time of an algorithm grows with the complexity of the input data. Here n is some measure of the size input data, and $f(n)$ denotes the number of "steps" an algorithm at worst needs for this size of input data.

For instance, there are various algorithms for deciding if a given number n is prime. The most naive algorithm is to test whether n is divisible by 2, then by 3, and so on up to $\sqrt{n}$. If we for simplicity treat each divisibility test as taking C steps for some fixed C to be counted, the run time of this algorithm is $C\sqrt{n}$, which is $O(\sqrt{n})$.

For comparison, the current best deterministic algorithm for testing if a number is prime, the so called AKS test, has a blazing fast running time of $O((\log n)^7)$.

Example. If $f(n) = \sum_{k=0}^{d} c_k n^k$ with $c_d \neq 0$ (so $f(n)$ is a polynomial of degree d), then $f(n) \in O(n^d)$. To see this, note that for $n \geq 1$, we have

$$\frac{|f(n)|}{n^d} = \frac{|\sum_{k=0}^{d} c_k n^k|}{|n^d|} \leq \frac{\sum_{k=0}^{d} |c_k||n^d|}{|n^k|} = \sum_{k=0}^{d} |c_k||n|^{k-d} \leq \sum_{k=0}^{d} |c_k|.$$

so we may take $C = \sum_{k=0}^{d} |c_k|$ in the definition.

Definition. Let $g\colon \mathbb{N} \to \mathbb{R}$. Then we define

$$\Omega(g(n)) = \{f(n)\colon \mathbb{N} \to \mathbb{R} \mid \text{ there exist } n_0, C > 0 \text{ such that } |f(n)| \geq C|g(n)| \text{ if } n \geq n_0\}$$

Remark. If $f(n) \in \Omega(g(n))$, this means that, up to multiplying by some constant, $f(n)$ grows at least as fast as $g(n)$.

We have $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$.

Example. If $f(n)$ is a polynomial of degree d, then $f(n) \in \Omega(n^d)$.

Definition. Let $g\colon \mathbb{N} \to \mathbb{R}$. Then

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n)).$$

Remark. Saying $f(n) \in \Theta(g(n))$ is saying that $f(n)$ grows, up to multiplying by constants, just as fast as $g(n)$. More precisely, it says that there exist $n_0, C_1, C_2 > 0$ such that for all $n \geq n_0$ we have

$$C_1|g(n)| \leq |f(n)| \leq C_2|g(n)|.$$

The relation $f(n) \in \Theta(g(n))$ is also written $f(n) \asymp g(n)$. The symbol $\asymp$ defines an equivalence relation, meaning

(1) for all $f(n)$, we have $f(n) \asymp f(n)$.
(2) if $f(n) \asymp g(n)$, then $g(n) \asymp f(n)$
(3) if $f(n) \asymp g(n)$ and $g(n) \asymp h(n)$, then $f(n) \asymp h(n)$.

The relation $f(n) \asymp g(n)$ means that $f(n)$ and $g(n)$ grow "at the same rate", in the sense that the fraction

$$\frac{|f(n)|}{|g(n)|}$$

lies within some fixed interval $[C_1, C_2]$ for all sufficiently large values of n. An even stronger relationship is the following.

Definition. Let $f(n), g(n)\colon \mathbb{N} \to \mathbb{R}$. We say that $f(n)$ and $g(n)$ are **asymptotically equal**, and write $f(n) \sim g(n)$, if

$$\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} = 1.$$

Remark. If $f(n) \sim g(n)$, then $f(n) \asymp g(n)$.

Remark. Just as with $\asymp$, the relation $\sim$ is an equivalence relation.

Example. For polynomials $p(n)$ and $q(n)$, we have the following interpretations of symbols $\prec, \asymp$ and $\sim$:

- $p(n) \prec q(n)$ if and only if $\deg p < \deg q$.
- $p(n) \asymp q(n)$ if and only if $\deg p = \deg q$.
- $p(n) \sim q(n)$ if and only if $\deg p = \deg q$ and the coefficient of the highest-degree term of $p(n)$ equals that of $q(n)$.

Example. Let $\pi(n)$ be the number of primes p with $1 \le p \le n$. Then the **prime number theorem**, proved by Hadamard and de la Vallée Poussin in 1896, says that

$$\pi(n) \sim \frac{n}{\log n}$$

0.2. **Asymptotics of factorials.** Most of the common functions we encounter are easy to sort by the relations described above. The factorial function $n!$ is more subtle. First, since

$$\frac{2^n}{n!} = \frac{2}{1} \times \frac{2}{2} \times \frac{2}{3} \times \cdots \times \frac{2}{n} \le \frac{2}{1} \times \frac{2}{2} \times \left(\frac{2}{3}\right)^{n-2} \to 0$$

as $n \to \infty$, we have $2^n \prec n!$. A similar argument shows for any $c \ge 1$ we have $c^n \prec n!$ as a lower bound for the growth of $n!$.

On the other hand, since

$$\frac{n!}{n^n} = \frac{n}{n}\frac{n-1}{n} \cdots \frac{1}{n} \le \left(\frac{n}{n}\right)^{\lceil n/2 \rceil} \left(\frac{n/2}{n}\right)^{\lfloor n/2 \rfloor} = 2^{-\lfloor n/2 \rfloor} \to 0$$

as $n \to \infty$, we get $n! \prec n^n$. So we have

$$c^n \prec n! \prec n^n.$$

We can improve on these estimates of $n!$ as follows. Estimating $n!$ is equivalent to estimating

$$\log n! = \sum_{i=1}^{n} \log i.$$

For any increasing real function $f(x)$, we have that (draw the graph of $f(x)$ and rectangles of height $f(i)$!)

$$\int_{j-1}^{k} f(x)dx \le \sum_{i=j}^{k} f(i) \le \int_{j}^{k+1} f(x)dx.$$

From this we get

$$\int_{1}^{n} \log(x)dx \le \sum_{i=2}^{n} \log i = \sum_{i=1}^{n} \log i \le \int_{1}^{n+1} \log(x)dx.$$

Since $\int \log(x)dx = x\log(x) - x + C$, we get from this that

$$n\log(n) - n + 1 \le \log(n!) \le (n+1)\log(n+1) - n.$$

Exponentiating these inequalities gives

$$e\left(\frac{n}{e}\right)^n \le n! \le e\left(\frac{n+1}{e}\right)^{n+1}. \tag{0.1}$$

We have

$$(n+1)^{n+1} = (n+1)(n+1)^n = (n+1)n^n \left(1+\frac{1}{n}\right)^n \leq e(n+1)n^n \tag{0.2}$$

using here

$$\left(1+\frac{1}{n}\right)^n \leq \left(\sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{n^k}\right)^n = \left(e^{\frac{1}{n}}\right)^n = e.$$

Combining (0.1) and (0.2) gives

$$e\left(\frac{n}{e}\right)^n \leq n! \leq (n+1)e\left(\frac{n}{e}\right)^n.$$

These inequalities are valid for all n. Asymptotically, we get an even more precise estimate by **Stirling's formula**[1]

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

[1]The same Stirling as the one with the numbers of the first and second kind.