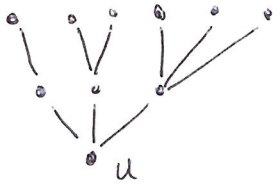
Trees (Sec. 7.1)

Def A graph is a **tree** if it is connected and has no circuits.

Ex



Can be drawn by choosing some vertex u as "root" & placing vertices v with $d(u,v) = h$ at height h

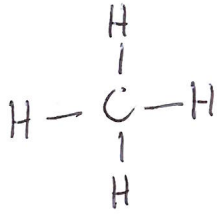
- Folder structures in computers
- Family trees

Ex Organic chemistry. Possible shapes of a molecule of 4 carbon atoms & 10 hydrogen atoms (C_4H_{10})

Consider as graph, $G=(V,E)$ assume

C has valency (= degree) 4

H has valency (= degree) 1

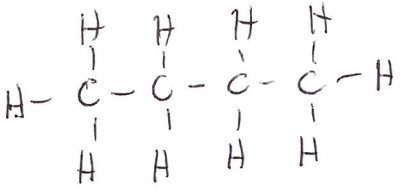


C_4H_4 methane

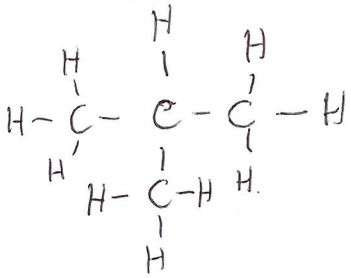
Know $|E| = \frac{\sum_{v \in V} d(v)}{2} = \frac{4\times 4 + 1\times 10}{2} = \frac{26}{2} = 13$

$|V| = 4+10 = 14.$

We'll see $|V| = |E| + 1 \implies G$ a tree, Two options



Butane



Isobutane

Def Let G be a connected graph. A spanning tree of G is a subgraph T of G which ① is a tree & ② has all vertices of G

Lemma Every connected graph has a spanning subtree.

Pf Let G be connected. If G has no circuits, G is tree, done. Else, let $k \in E$ be an edge in a circuit. k is not a cut edge (weekly problem), so $G' = G \setminus \{k\}$ is connected. Repeat this deletion process until no more circuits, have a spanning tree.

Thm Let $G = (V, E)$ be a graph. The following are equivalent

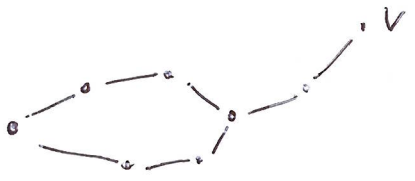
a) G is a tree

b) For $u, v \in V$, there is exactly one u,v-path

c) G is connected & $|V| = |E| + 1$.

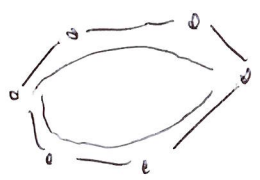
Pf a) $\Rightarrow$ b) G is a tree $\Rightarrow$ G is connected $\Rightarrow$ $\exists$ a u,v-path.

If there were two different paths, would get a circuit, contradicting G being a tree.



b) $\Rightarrow$ a) There is always a path $\Rightarrow$ G is connected.

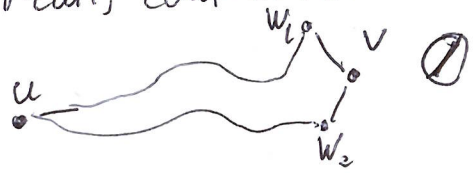
Assume for a contradiction G not a tree. Then G has a circuit. This gives two paths.



a) $\Rightarrow$ c) Claim: A tree has a vertex v of degree 1.

Proof: Choose some $u \in V$, & let $v \in V$ be such that $d(u,v)$ is as large as possible.

We claim $d(v)=1$. Otherwise v has 2 neighbours w_1, w_2 with $d(u,w_1) \leq d(u,w_2) \leq d(u,v)$. Shortest u,w_i-path does not pass through v, so get a circuit, contradiction.



①

Now assume $n=|V|$, let v have $d(v)=1$.

& let $G_1 = G \setminus \{v\}$.

Then G_1 is a tree with $|V(G_1)| = n-1$ & $|E(G_1)| = |E(G)|-1$.

Repeating get $G_2, G_3, \dots, G_{n-2}$, $|V(G_{n-2})| = 2$ $|E(G_{n-2})| = |E(G)| - (n-2)$.

Then G_{n-2} is $\bullet\!-\!\bullet$, so $|E(G_{n-2})| = 1$. Get $|V| - |E| = |V(G_{n-2})| - |E(G_{n-2})| = 1$.

c)$\Rightarrow$a) G has a spanning tree T. Have

$$|V(G)| = |V(T)|, \quad |E(T)| \leq |E(G)|, \text{ so}$$

$$1 = |V(G)| - |E(G)| \leq |V(T)| - |E(T)| = 1.$$

$$\Rightarrow |E(G)| = |E(T)| \Rightarrow G = T \Rightarrow G \text{ is a tree.}$$

Corollary: If G is a tree of order n

$$\sum_{u \in V} d(u) = 2|E| = 2(n-1) = 2n-2.$$

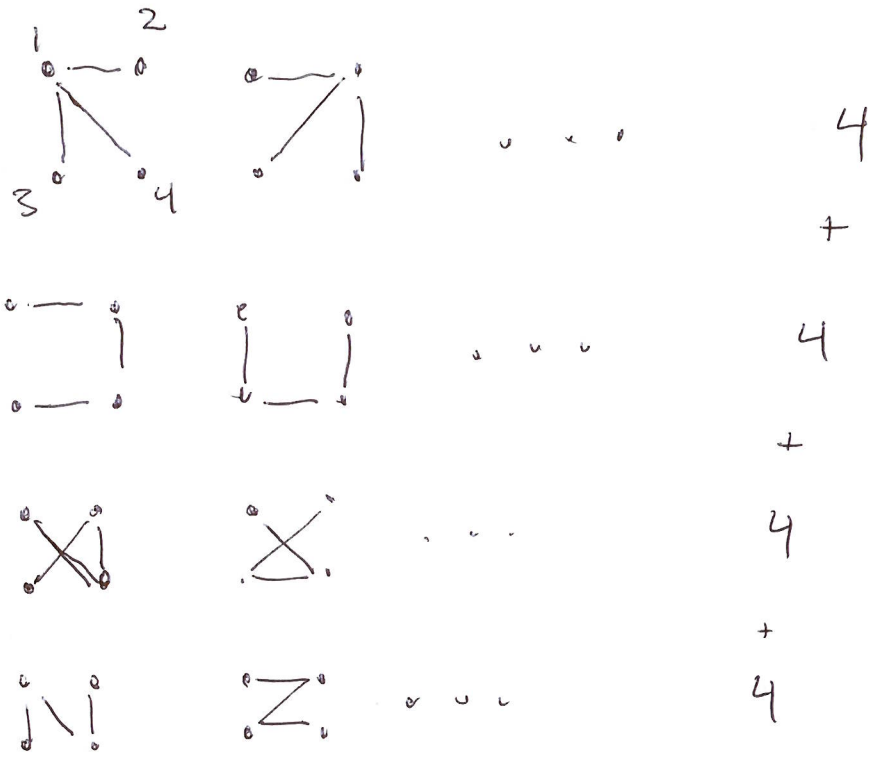
Question: How many spanning trees of a graph G?

General answer "Kirchhoff's theorem"

Special case $G = K_n$

		# spanning trees
K_1	1 $\bullet$	1
K_2	1 $\bullet\!-\!\bullet$ 2	1
K_3	2 / 1 $\bullet$ $\bullet$ 3	3

K_4



$= 16$ spanning trees

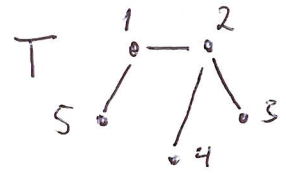
Thm (Cayley's formula) (Borchardt 1860)

The no. of spanning trees of K_n is $(n)^{n-2}$.

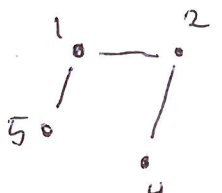
Proof by bijection to sequences $(a_1, a_2, \ldots, a_{n-2})$ $a_i \in \{1, \ldots, n\}$.

From spanning tree T to sequence ← Prüfer code of spanning tree

- List all labels of leaves in T
- Delete leaf with minimal label, set a_1 = neighbour of this leaf.
- Repeat with smaller tree to get $a_2, a_3, \ldots, a_{n-2}$.



$\{3,4,5\} \rightsquigarrow a_1 = 2$



$\rightsquigarrow a_2 = 2$

$\{4,5\}$



$\rightsquigarrow a_3 = 1$

$\{2,5\}$

Prüfer code $(2,2,1)$

From Prüfer code to tree:

- Observe, in Prüfer encoding, a vertex is listed once for each time a neighbour is deleted, & has one more edge for when it is deleted. So

$$d(i) = \#\text{times } i \text{ appears in } (a_1, _, a_{n-2}) + 1$$

- In particular, leaves of T are numbers not appearing in Prüfer code.
- First deletion must be of minimal elt. in $\{1, _, n\} \setminus \{a_1, _, a_{n-2}\}$, which must connect to a_1.

Given Prüfer code $(a_1, \ldots, a_{n-2})$, get tree by

- Let b_1 = minimal elt. of $\{1, \ldots, n\} \setminus \{a_1, _, a_{n-2}\}$
- Connect $b_1 a_1$
- Let b_2 = min. elt. of $\{1, _, n\} \setminus \{b_1, a_2, _, a_{n-2}\}$
- Connect $b_2 a_2$

⋮

- Let b_i = min. elt. of $\{1, _, n\} \setminus \{b_1, b_2, \ldots, b_{i-1}, a_{i+1}, _, a_{n-2}\}$
- Connect $b_i a_i$.

⋮

- Connect $b_{n-2} a_{n-2}$
- Let $\{b_{n-1}, b_n\} = \{1, \ldots, n\} \setminus \{b_1, _, b_{n-2}\}$.
- Connect $b_{n-1} b_n$.

Finding spanning trees in general