Matroids

Def A matroid is a pair $(S, \mathcal{U})$ where S is a finite set, $\mathcal{U} \subseteq 2^S$ (so elements of $\mathcal{U}$ are subsets of S) and

1. $\phi \in \mathcal{U}$
2. $A \subseteq B$ & $B \in \mathcal{U}$ implies $A \in \mathcal{U}$
3. $A, B \in \mathcal{U}$ & $|B| = |A|+1$ implies there is some $x \in B \setminus A$ so that $A \cup \{x\} \in \mathcal{U}$

If A set A lies in $\mathcal{U}$, say it is *independent*.

Ex Let $v_1, \dots, v_k \in \mathbb{R}^n$, $S = \{v_1, \dots, v_k\}$, say $A \subseteq S$ is independent if elements of A are linearly independent. This is matroid, check conditions

1. Obvious
2. Obvious
3. If $|A|+1 = |B|$ & $A, B \in \mathcal{U}$, then they span subspaces of dimension $|A|$ & $|B|+1$, respectively. B is not contained in $\text{Span}(A)$, so $\exists\, v \in B \setminus \text{Span}(A)$. Then $A \cup \{v\}$ is independent.

Prop In a matroid $(S, \mathcal{U})$, if $A, B \in \mathcal{U}$ are *maximal*, i.e. not contained in any bigger independent set, then $|A| = |B|$.

Pf Suppose not, then $|B| > |A|$ or $|A| > |B|$. By (2), can find $C \subseteq B$ with $|C| = |A|+1$ & $|C| = |A|+1$, By (3), can find $x \in C \setminus A$ s.t. $A \cup \{x\} \in \mathcal{U}$. Bah $|A \cup \{x\}| > |A|$, contradicting maximality of A.

Def The **rank** $r(M)$ of a matroid M is the size of a maximal independent subset.

Def A maximal independent subset is a **basis** of matroid.

Ex In $S = \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$ example $A \in \mathcal{U}$ is basis if it is a basis for Span (S).

Ex Can specify a matroid by listing bases, the set $\mathcal{U}$ is then determined by (2), since $A \in \mathcal{U} \iff A \subseteq B$ for some basis B.

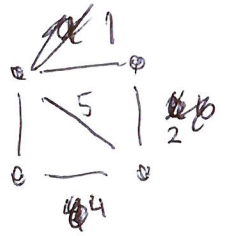
Ex $S = \{a, b, c, d\}$, bases $\{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}$ defines a matroid.

$S = \{a, b, c, d\}$, bases $\{a, b\}, \{c, d\}$ does not define a matroid, fails (3) with $A = \{a\}$, $B = \{c, d\}$.

So there is no collection of vectors $S = \{v_1, v_2, v_3, v_4\}$ where $\{v_1, v_2\}$ & $\{v_3, v_4\}$, but no other set of v_i, form a basis of Span (S).

Ex Let $G = (V, E)$ be a graph. Let $M = (E, W)$, where W is a set of subsets of E, and we say $A \in W \overset{\text{def}}{\iff} (E, A)$ is a forest (has no circuits)

Ex G



$E = \{1, 2, 3, 4, 5\}$

$W = 1, 2, 3, 4, 5, 12, 13, 14, 15,$
$23, 24, 25, 34, 35, 45,$
$123, 124, 234, 235, 245,$

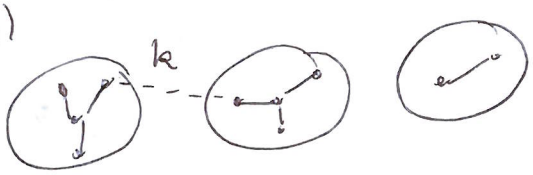
Thm $M=(E, \mathcal{W})$ is a matroid.

Pf ① & ② obvious.

③ Let $A \in \mathcal{W}$, so (V, A) is a forest. Assume (V, A) has a components. Each component is a tree, so has $|V|-|E|=1$. This gives

$$|V|-|A|=a \implies |A|=|V|-a.$$

If $B \in \mathcal{W}$ with $|B|=|A|+1$, then the number of components in (V, B) is $b=|V|-|B|=|V|-|A|-1=a-1$. This means (!) there is an edge $k \in B$ connecting 2 vertices in different components of (V, A).



Then $(V, A \cup \{k\})$ is a forest, so $A \cup \{k\} \in \mathcal{W}$ ($k \notin A$).

This gives ③. □

Matroid optimal basis: Let $M=(S, \mathcal{U})$ be a matroid, let $w: S \to \mathbb{R}$ be a function. Want a basis $\{a_1, \dots, a_r\}$ s.t. $\sum_{i=1}^{r} w(a_i)$ is minimal.

Greedy algorithm: ① Choose a_1 s.t. $w(a_1)$ is minimal.

② If $a_1, \dots a_i$ chosen, choose a_{i+1} among those s.t. $\{a_1, \dots, a_i, a_{i+1}\} \in \mathcal{U}$, pick a_{i+1} s.t. $w(a_{i+1})$ is minimal. Repeat until $\{a_1, \dots, a_r\}$ is a basis.

Thm The greedy algorithm is optimal: For any other basis $\{b_1, \dots, b_r\}$, have $\sum_{i=1}^{r} w(a_i) \le \sum_{i=1}^{r} w(b_i)$.

Pf Assume for a contradiction that $\sum_{i=1}^{r} w(b_i) < \sum_{i=1}^{r} w(a_i)$.

Arrange b_i so $w(b_1) \le w(b_2) \le \dots \le w(b_r)$. Then for some $i \in \{1, \dots, r\}$, $w(b_i) < w(a_i)$.

Now $\{a_1, a_2, -, a_{i-1}\}, \{b_1, b_2, -, b_i\} \in \mathcal{U}$, so by (3),

$\exists j \in \{1, -, i\}$ s.t. $\{a_1, -, a_{i-1}, b_j\} \in \mathcal{U}$ & $b_j \notin \{a_1, -, a_{j-1}\}$.

But $w(b_j) \leq w(b_i) < w(a_i)$, so at step i, algorithm would choose b_j over a_i. Contradiction! □

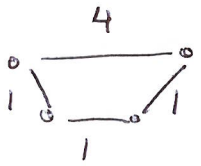
Corollary The greedy algorithm gives an optimal weighted spanning tree.

Shortest weighted paths, Dijkstra's algorithm

Let $G = (V, E)$ be a connected graph with weight function $w: E \to \mathbb{R}_{\geq 0}$.

For $u, v \in V$, and a u,v-path $P = u u_1 u_2 \cdots u_k v$, define its (weighted) length $l(P) = w(u u_1) + w(u_1 u_2) + \cdots + w(u_k v)$.

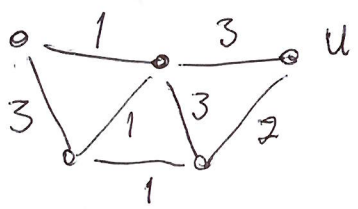
Let $d(u,v)$ be the minimal length of a u,v-path.

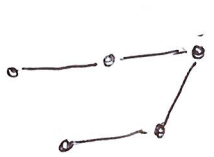
Ex: 

$d(u,v) = 3$.

Fix $u \in V$. Want spanning tree T in G such that moving in T gives minimal path from u to any vertex v.

Ex

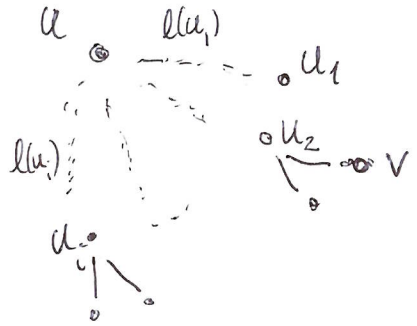




Dijkstra's algorithm Let $G = (V, E)$ be connected, $n = |V|$. Let $u \in V$. Build spanning tree by

① Set $u_1 = u$, $l(u_1) = 0$

(2) If vertices $u_1, \dots, u_i$ are added to tree, let $k_1, \dots, k_j$ be edges connecting a vertex u_ℓ to some v not in tree. Let $f(k_m) = w(k_m) + l(u_j)$, Choose k_m s.t. $f(k_m)$ is minimal, set $u_{i+1} = v$, $l(u_{i+1}) = f(k_m)$. If $i+1 = n$, done, else repeat step 2.



← Add edge $u_j v$ so $l(u_j) + w(u_j, v)$ minimal.

Exercise: Show that if $w(k) = 1 \ \forall k \in E$, then Dijkstra's algorithm is a breadth-first search.

~~Proof that Dijkstra's algorithm has the claimed property:~~
Let l be the length from u to v moving in T. Claim is $l = d(u,v)$.
Let $P = u u_1 \dots u_k v$ be a path of minimal length $d(u,v)$.
~~If P lies in the tree, claim holds. Otherwise~~
~~We claim that~~
Assume for a contradiction

Prop Let T be spanning tree produced by Dijkstra's algorithm. For any $v \in V$, the path in T from u to v has length $d(u,v)$ (so minimal).

Proof Assume for a contradiction that there is a v s.t. this fails, and let v be such that $d(u,v)$ is minimal* for v with this property. Then Let $P = u u_1 \dots u_k v$ be a minimal path in G. Then since $d(u,v)$ is minimal*, the distance path in T from u to u_k has length $d(u, u_k)$.

The edge argument

Let T be the spanning tree produced by Dijkstra's algorithm. For $v \in V$, let $d_T(u,v)$ be the weighted length of the unique u,v-path in T.

Prop For all $v \in V$, $d_T(u,v) = d(u,v)$

Proof Let $u_1, u_2, \ldots, u_n$ be the vertices of G in the order added by D's algorithm.

Claim: For every i, $1 \le i \le n$, we have

① $d(u, u_i) = d_T(u, u_i)$

② If $v \in V$ is such that $d(u,v) < d(u, u_i)$, then $v = u_j$ for some j, $1 \le j < i$.

Proof of claim by induction on i:

Clear for $i = 1$. Assume true for i, prove for $i+1$.

① If $d_T(u, u_{i+1}) >$