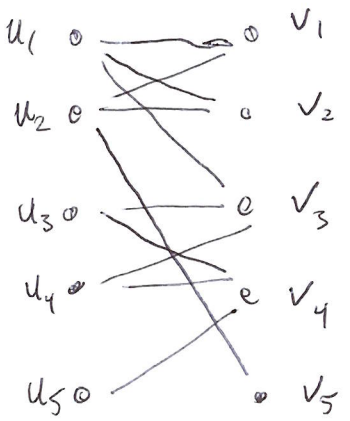
Bipartite graphs and matchings (Sec. 8.1 - 8.2)

Ex Consider a list of people $u_1, \dots, u_5$ & a list of presents $v_1, \dots, v_t$. Each person wants some of the presents, Can we assign presents to people so that every person gets one they want?

(*)



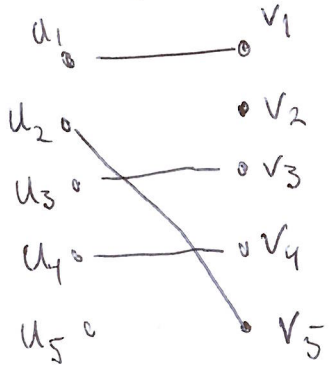
Represent by bipartite graph $(S \cup T, E)$ with $S = \{u_1, \dots, u_5\}$ $T = \{v_1, \dots, v_t\}$ & $u_i v_j \in E$ if person u_i wants present v_j.

Here impossible, since u_3, u_4, u_5 only accept v_3, v_4 ...

Def If $G = (S \cup T, E)$ is bipartite graph, a **matching** is a set $M \subseteq E$ such that no two k_1, k_2 in M are incident.

The **matching number** $m(G)$ is the biggest possible size of a matching, & a matching M with $|M| = m(G)$ is called a maximum matching.

Ex In (*), have $m(G) = 4$, a maximum matching M is



Remark Always have $m(G) \leq |S|$, $m(G) \leq |T|$.

Def For $A \subseteq S$, let $N(A) = \{ v \in T \mid uv \in E \text{ for some } u \in A \}$

$$= \bigcup_{u \in A} N(u).$$

Ex In ⊛ $N(\{u_3, u_4, u_5\}) = \{v_4, v_5\}$.

Thm (Hall's marriage theorem) Let $G = (S+T, E)$ be a bipartite graph. Then $m(G) = |S|$ if and only if, for all $A \subseteq S$, $|A| \leq |N(A)|$.

Pf Only if: Let $S = \{u_1, \ldots, u_s\}$. If M is a matching s.t. $|M| = |S|$, we have $M = \{u_1v_1, u_2v_2, \ldots, u_sv_s\}$ where the v_i are distinct & $u_iv_i \in E$ for all i. If $A = \{u_{i_1}, u_{i_2}, \ldots, u_{i_a}\}$, then $\{v_{i_1}, v_{i_2}, \ldots, v_{i_a}\} \subseteq N(A)$, so $|N(A)| \geq |A|$.

If: Assume $|A| \leq |N(A)|$ for all $A \subseteq S$. Assume for a contradiction that $m(G) < |S|$, & let M be a max. matching. We will build a bigger matching, giving contradiction.

Since $|M| < |S|$, have some u_0 unmatched. Since $|\{u_0\}| \leq |N(u_0)|$, u_0 has some neighbour, call this v_1

If v_1 is unmatched, add u_0v_1 to M, we win.

Otherwise, let $u_1 \in S$ be s.t. $u_1v_1 \in M$.

Since $|\{u_0, u_1\}| \leq |N(\{u_0, u_1\})|$, can find $v_2 \in N(\{u_0, u_1\})$.

If v_2 unmatched, stop, otherwise repeat. Let u_2 be s.t. $u_2v_2 \in M$, & repeat.

at step i, have $u_0, \dots, u_i \in S$, $v_1, v_2, \dots, v_i \in T$, since

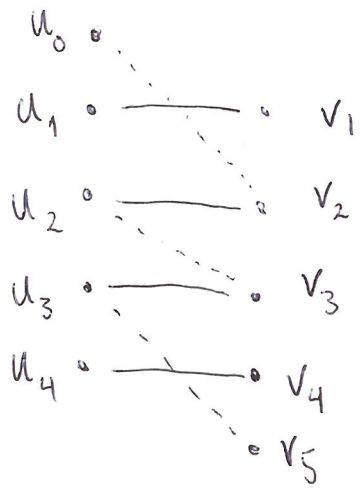
$$i+1 = |\{u_0, \dots, u_i\}| \leq |N(\{u_0, \dots, u_i\})|,$$

so can find $v_{i+1} \in N(u_0, \dots, u_i) \setminus \{v_1, \dots, v_i\}$.

At some point find v_r unmatched, have sequences

$u_0, u_1, u_2, \dots, u_{r-1} \in S$ $v_1, v_2, \dots, v_r \in T$, with properties

- $u_1 v_1, u_2 v_2, \dots, u_{r-1} v_{r-1} \in M$
- u_0, v_r unmatched
- For all v_i, there is some $j < i$ s.t. $u_j v_i \in E$.



Can find path from v_r to u_0 where every other edge is in M

$v_r u_{i_1} v_{i_1} u_{i_2} v_{i_2} \dots v_{i_\ell} u_0$

$r > i_1 > i_2 > \dots > i_\ell$

Change M by dropping $u_{i_1} v_{i_1}, u_{i_2} v_{i_2}, \dots u_{i_\ell} v_{i_\ell}$,
& adding $v_r u_{i_1}, v_{i_1} u_{i_2}, v_{i_2} u_{i_3}, \dots v_{i_\ell} u_0$.
This gives bigger matching. □

The condition

$$\forall A \subseteq S \quad |A| \leq |N(A)|$$

is equivalent to there being a matching of size $|S|$.
If this fails, we can still use the number $|A| - |N(A)|$ to say how big a max'l matching is:

Thm Let $G=(S+T, E)$ be bipartite. Then

$$m(G) = |S| - \max_{A \subseteq S} \left(|A| - |N(A)| \right).$$

Pf Prove as $\leq$ and $\geq$

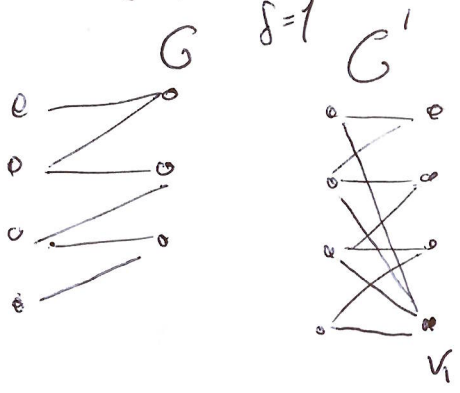
$\leq$: $m(G) \leq |S| - \max_{A \subseteq S} (|A| - |N(A)|)$ holds if

$m(G) \leq |S| - (|A| - |N(A)|)$ holds for all $A \subseteq S$.

Given a max'l matching M, at most $|N(A)|$ of the vertices in A are incident to an edge in M. Thus

$$m(G) = |M| \leq |S \setminus A| + |N(A)| = |S| - (|A| - |N(A)|).$$

$\geq$: Let $\delta = \max_{A \subseteq S} |A| - |N(A)|$. Make a new bipartite graph $G' = (S + T', E')$, where $T' = T \cup \{v_1, v_2, \ldots, v_\delta\}$, & every v_i is connected to every $u \in S$



In G', every nonempty $A \subseteq S$ has δ extra neighbours, so $|A| \leq |N(A)|$ holds.

Take a matching M' in G' with $|M'| = |S|$.

Restricting to G gives matching M of size $|M| \geq |M'| - \delta = |S| - \delta$, as claimed.

Various applications of the marriage theorem

Thm Let $A_1, A_2, \ldots, A_n$ be finite sets. Then there exists a way to pick distinct elements $x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n$ if and only if, for each $I \subseteq \{1, \ldots, n\}$, we have

$$(*) \quad \left|\bigcup_{i \in I} A_i\right| \geq |I|$$

Pf Take bipartite graph $(S+T, E)$ with $S = \{1, 2, \ldots, n\}$, $T = \bigcup_{i=1}^{n} A_i$, & an edge ix $(i \in \{1, \ldots, n\}, x \in \bigcup A_i)$ if $x \in A_i$. Then a matching is a sequence $x_1 \in A_1, \ldots$ & $(*)$ is condition in Hall's theorem.

Vertex covers: Let $G = (S+T, E)$ be a bipartite graph. A vertex cover of G is a $D \subseteq S \cup T$ such that every edge in G meets D.

If D is a vertex cover & M a matching, then $|D| \geq |M|$, so $\min_D |D| \geq m(G)$. But, in fact

Thm $\min_D |D| = m(G)$

Pf $m(G) = |S| - \max_{A \subseteq S} (|A| - |N(A)|)$. Let $A \subseteq S$ be such that $m(G) = |S| - |A| + |N(A)|$.
Then $D = S \setminus A \cup N(A)$ is a vertex covering of size $m(G)$.

Example: Let B be a square matrix of size $(n \times n)$. A diagonal of B is a choice of n entries in the matrix, no two in the same row or column

$n = 3$ diagonals

$$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \begin{pmatrix} & * & \\ & & * \\ * & & \end{pmatrix} \begin{pmatrix} & & * \\ * & & \\ & * & \end{pmatrix} \begin{pmatrix} * & & \\ & & * \\ & * & \end{pmatrix} \begin{pmatrix} & * & \\ * & & \\ & & * \end{pmatrix} \begin{pmatrix} & & * \\ & * & \\ * & & \end{pmatrix}.$$

Prop The maximal number of non-zero entries in a diagonal of B = the minimal number of rows & columns such that every non-zero entry of B

Claim: If d is the max'l number of non-zero entries you can get in a diagonal of B, then you can find e rows, f columns, with $d = e + f$, so that every non-zero entry of B lies in one of the rows or one of the columns.

Proof Make graph $G = (S+T, E)$, $S = \{u_1, \ldots, u_n\}$, $T = \{v_1, \ldots, v_n\}$, $u_i v_j \in E$ if $B_{ij} \neq 0$.

A maximal matching M gives a diagonal with $|M|$ non-zero entries.

A vertex covering is a choice of e rows & f columns so all non-zero entries lie in one of the rows or columns.