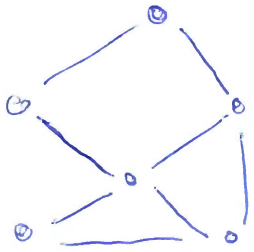
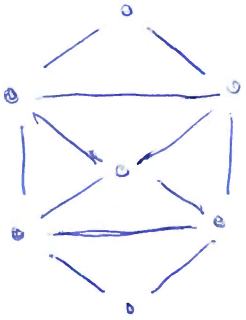
Eulerian graphs & the travelling salesman problem

Let $G = (V, E)$ be a graph. An Euler circuit is a circuit in G containing every edge of G once

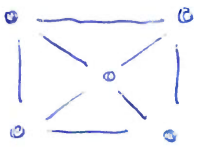
An Euler path or open Euler circuit is a path in G containing every edge once



Euler path



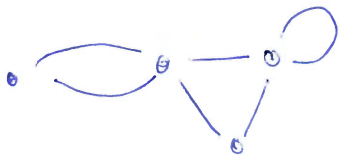
Euler circuit



Neither

Def A multigraph is a set of vertices V & a set of edges E, where ① each pair of vertices can now have multiple edges between them and ② a vertex can have a self-edge.

Multigraph:



Exercise:

Formally encode as triple $G = (V, E, f)$ where V, E finite sets $f: E \to \binom{V}{2} \cup V$.

Definition of Euler circuit/path extends to multigraph.

Def A multigraph is Eulerian if it has an Euler circuit.

Thm A multigraph G is Eulerian $\iff$ G is connected & every vertex has even degree.

Pf $\Rightarrow$ An Euler circuit C shows that G is connected.
Since C "uses up" 2 edges incident to u each time it passes u, find $d(u)$ even for all $u \in V$.

$\Leftarrow$ Argue by induction on $q = |E|$.
Choose a circuit C in G with distinct edges, of max'l length.
If $E(C) = E$, then C is Eulerian & so G is.
Otherwise (*), let $G' = (V, E \setminus E(C))$, & let $u \in V$ be such that not every edge incident to u is in C. Since C has an even no. of edges thru each vertex, every vertex has even degree in G'.
Let G'' be the component of G' containing u. Then by induction on q, G'' has an Eulerian circuit C'.
Adding C' to C gives bigger circuit than C, contradicting (*). □

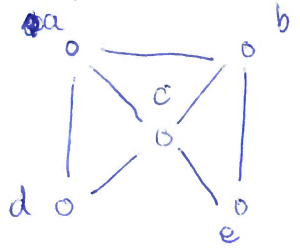
Variant: A multigraph G has Eulerian path $\iff$ G is connected & has 0 or 2 vertices of odd degree.

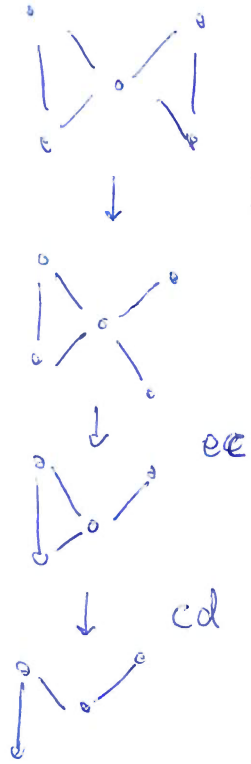
Algorithm for finding Eulerian path: If 0 vertices of odd degree, (so want Eulerian circuit), start in any vertex u.
Otherwise, start in one of two odd-degree vertices.
- If current vertex is u, find some edge uv which is not a cut edge (always possible!)

- Add edge uv, erase uv from graph. Current vertex is v. Repeat.
  to path

Example

- Odd-degree vertices a, b
- Choose a as starting vertex.



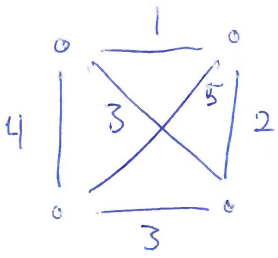


Path is abecdacb

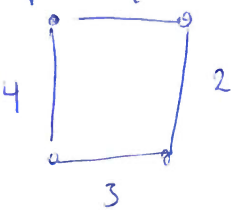
Travelling salesperson problem (TSP)

Given n cities, a travelling salesperson wants to travel through every city exactly once, returning to start city, with minimal travel distance.

Ex

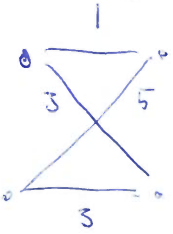


optimal



$1+2+3+4=10$

not optimal



$1+3+3+5=12$

Def A Hamiltonian circuit in a graph G is a circuit C such that $V(C)=V(G)$, i.e. passing through every vertex once.

TSP, formal statement: Consider $K_n = (V, E)$ $|V|=n$, $|E|=\binom{n}{2}$,

let $w: E \to \mathbb{R}_{\geq 0}$ be a function.

Find the Hamiltonian circuit C such that

$$w(C) = \sum_{k \in E(C)} w(k) \text{ is minimal.}$$

Matrix reformulation: Let (w_{ij}) be $\{u_1, \dots, u_n\} = V$, let

$w_{ij} = w(u_i u_j)$ if $i \neq j$

$w_{ii} = 0 \quad \forall\, i.$

So (w_{ij}) is symmetric non-negative matrix, want fixed-point free permutation π of $\{1, 2, \dots, n\}$ minimising $\sum_{i=1}^{n} w_{i\pi(i)}$.

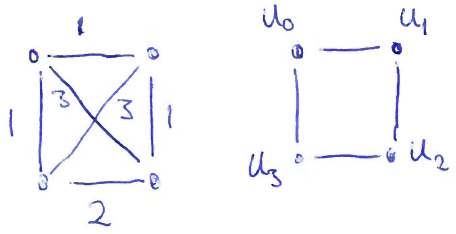
Nearest-neighbour heuristic

Pick vertex $u_0 \in V$ at random.

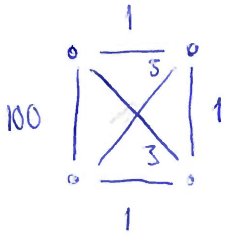
If vertices $u_0, u_1, \dots, u_i$ are picked, choose u_{i+1} among $V \setminus \{u_0, \dots, u_i\}$ such that $w(u_i u_{i+1})$ is as small as possible.

Repeat until Hamiltonian circuit constructed.

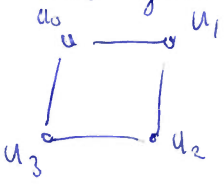
Ex:



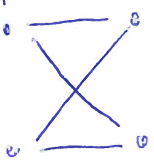
NN can do poorly:



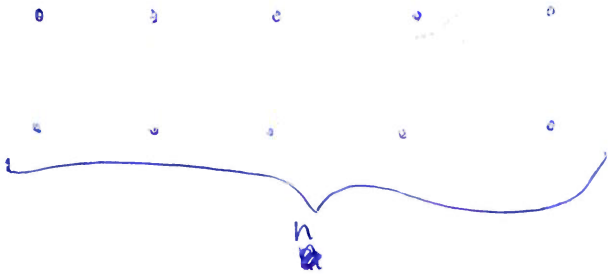
NN gives



Optimal is



K_{2n}, $w_{euc}(u, v)$ = distance in plane



Optimal is



NN can give



etc