Latin squares (12.3)

Motivation: You research fertilisers A, B, C, D. Test these in 4×4 plots of land with 4 different soil types & 4 levels of irrigation. How to arrange planting so every fertiliser tested in every soil type & with every irrigation?

Soil \ Irrigation	1	2	3	4
1	A	B	C	D
2	B	C	D	A
3	C	D	A	B
4	D	A	B	C

Each letter appears once in every row & column.
If effects of soil type & irrigation are independent, this gives efficient experiment.

Def Let S be an n-set. A Latin square of order n over S is a function $L: \{1, _, n\} \times \{1, 2, _, n\} \to S$ such that

$L(i,j) \neq L(i,j')$ if $j \neq j'$ & $L(i,j) \neq L(i',j)$ if $i \neq i'$.

Ex. $L(1,1) = A$, $L(1,2) = B$, _ is the Latin square above.

Ex Let $S = \{0, 1, _, n-1\}$. Then $L(i,j) = i+j \mod n$ gives a Latin square

2	0	1
0	1	2
1	2	0

2	3	0	1
3	0	1	2
0	1	2	3
1	2	3	0

Ex A completed Sudoku puzzle gives a Latin square of order 9, (but not vice versa).

Orthogonal Latin squares

Now want to run the experiment twice, & want that every pair of fertilisers is tested on some plot. (fertiliser)

Equivalently, no plot should see the same sequence of fertilisers.

Def Two Latin squares $L_1, L_2 : \{1,\dots,n\}\times\{1,\dots,n\} \to A$ are orthogonal if there is no $(i,j)\neq(i',j')$ such that $L_1(i,j)=L_2(i',j')$ and $L_2(i,j)=L_2(i',j')$.

Equivalently, for every $a_1, a_2 \in A$, there is exactly one $(i,j)\in\{1,\dots n\}^2$ s.t. $L_1(i,j)=a_1$, $L_2(i,j)=a_2$.

Ex

0	1	2
1	2	0
2	0	1

&

0	2	1
1	0	2
2	1	0

are orthogonal

0	1	2	
1	2	0	
2	0	1	

&

0			
2			

A	B	C	D
B	C	D	A
C	D	A	B
D	A	B	C

&

A	D	C	B
B	A	D	C
C	B	A	D
D	C	B	A

are not (A,A) appears twice.

— " — &

A	C	D	B
B	D	A	C
C	A	B	D
D	B	C	A

are not orthogonal. (AB) appears twice

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

&

A	B	C	D
C	D	A	B
D	C	B	A
B	A	D	C

are orthogonal.

A set $\{L_1,\dots,L_m\}$ of Latin squares of order n are mutually orthogonal if for every a,b L_a is orthogonal to L_b.

Question: What is the maximal size of a set of mutually orthogonal Latin squares (MOLS) of order n?

Call this $N(n)$.

Have shown $N(n)\geq 1$ $\forall n$ (There is at least one Latin square of order n)

Have seen $N(3), N(4) \geq 2$.

Euler's conjecture (1782): $N(6) = 1$.

More generally if $n \equiv 2 \pmod 4$, then $N(n) = 1$ $(1 = N(2) = N(6) = N(10) = \cdots$

Tarry (1901): $N(6) = 1$ (checks 9408 order 6 Latin squares).

MacNeish (1921): Claims proof of $1 = N(6) = N(10) = \cdots$

Bose-Parker-Shrikhande (1959) Prove $2 \leq N(10), N(14), N(18), \ldots$

Exact value of $N(10)$ unknown still.

Theorem If $n = p^k$ with p a prime, then we can find $n-1$ mutually orthogonal Latin squares.

Pf Let $A = GF(p^k) = \{0, a_1, a_2, \ldots, a_{n-1}\}$. For $h = 1, 2, \ldots, n-1$, define a Latin square

$$L_h : A \times A \longrightarrow A$$

by

$$L_h(x, y) = a_h x + y.$$

Claim 1: These are Latin squares.

If $y \neq y'$, L_h

$$(a_h x + y) - (a_h x + y') = y - y' \neq 0,$$

so $(a_h x + y) \neq a_h x + y$

If $x \neq x'$, $(a_h x + y) - (a_h x' + y) = a_h (x - x') \neq 0$ $(a_h \neq 0)$ & $GF(p^k)$ is field.

Claim 2: If $h \neq h'$, then L_h & $L_{h'}$ orthogonal:

Given $(\alpha, \beta) \in A^2$, want $L_h(x, y) = \alpha$ $L_{h'}(x, y) = \beta$

to happen for one pair $(x, y) \in A^2$. This is

$$\begin{aligned} a_h x + y &= \alpha \\ a_{h'} x + y &= \beta \end{aligned} \iff \begin{aligned} (a_h - a_{h'}) x &= \alpha - \beta \\ a'_h x + y &= \beta \end{aligned} \iff \begin{aligned} x &= \frac{\alpha - \beta}{a_h - a_{h'}} \\ y &= \beta - \frac{a_{h'}(\alpha - \beta)}{a_h - a_{h'}}. \end{aligned}$$

□

Thm For all

Ex: $n=3$, $A=GF(3)=\{0,1,2\}$

L_1	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(x = row, y = column)

L_2	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

$L_1(x,y)=1x+y$ $\qquad$ $L_2(x,y)=2x+y$

Thm For any $n\geq 2$, $N(n)\leq n-1$.

Pf ↯ Assume for contradiction we have n MOLS

$L_1, L_2, \ldots, L_n$.

May assume that $L_i(1,j)=j$ $\forall i,j\in\{1,\ldots,n\}$,

if not, define $\pi_i:\{1,\ldots,n\}\to\{1,\ldots,n\}$ by $\pi_i=L_i(1,j)$ &

replace L_i by $\pi_i^{-1}\circ L_i$.

$L_1(2,1)=k$, then must have $L_i(2,1)\neq k,1$ $\forall i\geq 2$.

Must have $i\neq i'$ s.t. $L_i(2,1)=L_{i'}(2,1)$, but

this contradicts orthogonality of L_i & $L_{i'}$. ↯ □

Thm (MacNeish bound) For $n_1,n_2\geq 2$, have $N(n_1n_2)\geq \min(N(n_1),N(n_2))$.

Pf Equivalent formulation: Given k MOLS of order n_1 & of order n_2, must have k MOLS of order n_1n_2.

Given MOLS $L_1,\ldots,L_k: A_1\times A_1\to A_1$ & $L_1',\ldots,L_k': A_2\times A_2\to A_2$,

with $|A_1|=n_1$ & $|A_2|=n_2$. Let $A=A_1\times A_2$, define

$L_i'': A^2\to A$ by $L_i''((x,y),(z,w))=(L_i(x,z),L_i'(y,w))$.

Check these are MOLS.