27th March, 2025

MAT3250

Mandatory assignment 1 of 1

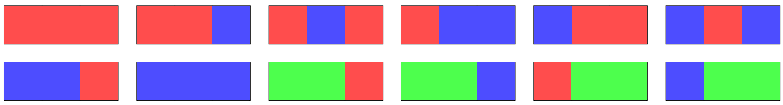
Problem 1. Consider a $1 \times n$ rectangle which is to be covered by the following three kinds of blocks:

- red 1×1-blocks
- blue 1×1-blocks
- green 1×2-blocks

The following figure illustrates all ways of doing this for $n = 1, 2, 3$:

$n = 1$: 

$n = 2$: 

$n = 3$: 

Let $a_0 = 1$, and for $n \geq 1$, let a_n be the number of coverings as above.

(a) Find a recurrence relation for a_n in terms of a_{n-1} and a_{n-2} for all $n \geq 2$.

(b) Express the generating function of the sequence a_n as a rational function.

(c) Find an integer k and real numbers $b_1, \ldots, b_k$, $c_1, \ldots c_k$ such that

$$a_n = \sum_{i=1}^{k} c_i b_i^n$$

holds for all $n \geq 0$.

(d) For which numbers $b \in \mathbb{R}_{>0}$ is it true that the function $f(n) = a_n$ lies in $O(b^n)$?

Solution:

(a) Let $n \geq 2$, we want to determine a_n. There are three choices for the rightmost piece in the covering, namely red, blue and green. If the rightmost piece is red, there is a $1 \times (n-1)$-rectangle left to cover, which shows that there are a_{n-1} ways to cover a rectangle where the rightmost piece is red. Similarly, there are a_{n-1} ways to cover the rectangle with a blue rightmost piece, and a_{n-2} ways to cover the rectangle with a green rightmost piece. This shows that

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} = 2a_{n-1} + a_{n-2}.$$

(b) Let $A(z) = \sum_{n\geq 0} a_n z^n$. The recursion relation shows that

$$\sum_{n\geq 2} a_n z^n = \sum_{n\geq 2} (2a_{n-1} + a_{n-2}) z^n = \sum_{n\geq 1} 2a_n z^{n+1} + \sum_{n\geq 0} a_n z^{n+2}.$$

The left hand side of this equation is $A(z) - a_0 - a_1 z$, while the right hand side is $2z(A(z) - a_0) + z^2 A(z)$. Inserting $a_0 = 1$ and $a_1 = 2$, we find the equation

$$A(z) - 1 - 2z = 2z(A(z) - 1) + z^2 A(z),$$

which we solve to get

$$A(z) = \frac{1}{1 - 2z - z^2}.$$

Alternatively: By Theorem 3.1. of Aigner, we know that $A(z) = p(z)/q(z)$, where $q(z) = 1 - 2z - z^2$ and the degree of $p(z)$ is at most 1. Multiplying out, we find

$$p(z) = q(z)A(z) = (1-2z-z^2)(a_0+a_1 z+\sum_{n\geq 2} a_n z^n) = (1-2z-z^2)(1+2z+\sum_{n\geq 2} a_n z^n).$$

Since we know $p(z)$ is a polynomial of degree at most 1, we only need to compute the terms of z-degree at most 1 in this product, which gives $p(z) = 1 - 2z + 2z = 1$. Hence $A(z) = p(z)/q(z) = 1/(1 - 2z - z^2)$.

(c) We want to use partial fraction decompositions to write

$$\frac{1}{1 - 2z - z^2} = \frac{c_1}{1 - b_1 z} + \frac{c_2}{1 - b_2 z}.$$

We know that then b_1, b_2 are the roots of the polynomial $w^2 - 2w - 1$. Solving with the quadratic formula gives

$$b_1 = 1 + \sqrt{2}, \qquad b_2 = 1 - \sqrt{2}$$

Multiplying both sides by $(1 - b_1 z)(1 - b_2 z)$ we get

$$1 = c_1(1 - b_2 z) + c_2(1 - b_1 z) = (c_1 + c_2) - (b_1 c_2 + b_2 c_1) z.$$

To find c_1 and c_2, we either solve the system of equations

$$1 = c_1 + c_2$$
$$0 = c_1 b_2 + c_2 b_1,$$

or alternatively argue as follows: Inserting $z = b_1^{-1}$ and $z = b_2^{-1}$ gives

$$1 = c_1(1 - b_1 b_2^{-1})$$

and

$$1 = c_2(1 - b_1 b_2^{-1}).$$

This gives

$$c_1 = (1 - b_1 b_2^{-1})^{-1} = \frac{1}{2} - \frac{\sqrt{2}}{4}$$

and

$$c_2 = (1 - b_2 b_1^{-1})^{-1} = \frac{1}{2} + \frac{\sqrt{2}}{4}.$$

In terms of the generating function, we then have

$$\sum_{n=0}^{\infty} a_n z^n = \frac{1}{1 - 2z - z^2} = \frac{c_1}{1 - b_1 z} + \frac{c_2}{1 - b_2 z} = \sum_{n=0}^{\infty} (c_1 b_1^n + c_2 b_2^n) z^n,$$

so

$$a_n = c_1 b_1^n + c_2 b_2^n$$

for all $n \geq 0$.

(d) We have $f(n) \in O(b^n)$ if and only if there is some constant $C > 0$ and integer N such that $|f(n)| \leq C b^n$ for all $n \geq N$. We claim that $f(n) \in O(b^n)$ if and only if $b \geq b_1$.

We first show that $f(n) \in O(b_1^n)$. Since $|b_2| < |b_1|$, we have

$$|c_2 b_2^n| < |c_2 b_1^n| = |c_2||2b_1^n|$$

for all $n \geq 0$. Then

$$|f(n)| = |c_1 b_1^n + c_2 b_2^n| \leq |c_1 b_1^n| + |c_2 b_2^n| \leq |c_1 b_1^n| + |c_2 b_1^n| = (c_1 + c_2) b_1^n$$

for all n. We may thus take $C = c_1 + c_2$ in the definition, showing $f(n) \in O(b_1^n)$. Since $b_1^n \in O(b^n)$ for any $b \geq b_1$, it follows that $f(n) \in O(b^n)$ for all $b \geq b_1$.

We now show that if $0 < b < b_1$, then $f(n) \notin O(b^n)$. We have

$$|f(n)|/b^n = |c_1(b_1/b)^n + c_2(b_2/b)^n| = c_1(b_1/b)^n |1 + c_2(b_2/b_1)^n|.$$

Since $|b_2| < |b_1|$, we have $(b_2/b_1)^n \to 0$ as $n \to \infty$, and so $|1 + c_2(b_2/b_1)^n| \to 1$ as $n \to \infty$. Since $b < b_1$, we have $(b_1/b)^n \to \infty$ as $n \to \infty$. It follows that $|f(n)|/b^n \to \infty$ as $n \to \infty$, which means there can be no C such that $|f(n)|/b^n \leq C$ for sufficiently large n. Hence $f(n) \notin O(b^n)$.

Problem 2. What is the number of permutations π of the set $\{1, 2, \ldots, n\}$ such that none of $1, 2$ or 3 are fixed points of π?

Solution: Let A be the set of permutations of $\{1, \ldots, n\}$, and for $i \in \{1, \ldots, n\}$, let $A_i \subseteq A$ be the subset of permutations such that i is a fixed point. We are asked to find the size of the set of permutations such that none of $1, 2, 3$ are fixed points, which is the set $A \setminus (A_1 \cup A_2 \cup A_3)$.

By the inclusion exclusion principle, we have

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Permutations in A_1 fix the element 1, and permute the elements $2, \ldots, n$. Therefore A_1 is in bijection with the set of permutations of $\{2, \ldots, n\}$, so $|A_1| = (n-1)!$. Similarly $|A_2| = |A_3| = (n-1)!$.

Permutations in $A_1 \cap A_2$ fix the elements 1 and 2, and permute the elements $3, \ldots, n$, giving similarly $|A_1 \cap A_2| = (n-2)!$. Similarly we find $|A_2 \cap A_3| = |A_1 \cap A_3| = (n-2)!$ and $|A_1 \cap A_2 \cap A_3| = (n-3)!$.

Inserting this we find our answer

$$|A \setminus (A_1 \cup A_2 \cup A_3)| = |A| - |A_1 \cup A_2 \cup A_3| = n! - 3(n-1)! + 3(n-2)! - (n-3)!.$$

Problem 3. Let $G = (V, E)$ be a graph. A circuit of length l in G is a subgraph C of G with l distinct vertices $u_1, u_2, \ldots, u_l$ and l distinct edges $u_1u_2, u_2u_3, \ldots, u_{l-1}u_l, u_lu_1$. A **Hamiltonian circuit** of G is a circuit containing every vertex of G.

(a) Find all Hamiltonian circuits in the hypercube graph Q_3 that contain the edges $(0,0,0)(0,0,1)$ and $(0,0,1)(0,1,1)$.

(b) Find all Hamiltonian circuits in the complete bipartite graph $K_{3,3}$.

(c) Show that $K_{3,4}$ has no Hamiltonian circuit.

(d) For $n \geq 3$, find the number of Hamiltonian circuits in K_n.

Solutions: a): We denote vertex (i, j, k) as ijk and edges by the symbol $-$. We consider our options for how to extend the path $000-001-011$ one vertex at a time. Having built the start of the circuit $000-001-011-u_2-\cdots-u_i$, let us say the *valid* choices for the successor of u_i are vertices v such that v is a neighbour of u_i, and such that v has not appeared previously in the path.

Beginning with $000 - 001 - 011$, the valid successors of 011 are (A) 010 or (B) 111.

Case A: With the path $000 - 001 - 011 - 010$, note first that 100 is the only one valid choice of vertex preceeding 000, so our circuit must contain the path $100 - 000 - 001 - 011 - 010$. From this point on, there is only one valid successor of 010 and all the vertices added after it, leading to the Hamiltonian circuit

$$100 - 000 - 001 - 011 - 010 - 110 - 111 - 101 - 100$$

Case B: Our path is $000 - 001 - 011 - 111$, and we have two valid successors of 111, namely (B1) 101 and (B2) 110.

Case B1: Our path is $000 - 001 - 011 - 111 - 101$. From this point there only one valid successor of 101 and all vertices added after it, giving Hamiltonian circuit

$$000 - 001 - 011 - 111 - 101 - 100 - 110 - 010 - 000.$$

Case B2: Our path $000 - 001 - 011 - 111 - 110$. The two valid choices of successor of 110 are (B2a) 100 and (B2b) 010.

Case B2a: Our path is $000 - 001 - 011 - 111 - 110 - 100$. The only valid successor of 100 is 101, but this then has no valid successors, so we do not find a Hamiltonian circuit.

Case B2b: Our path is $000 - 001 - 011 - 111 - 110 - 010$. The vertex 010 has no valid successors, so we do not find a Hamiltonian circuit.

b) Let us say that the two vertex sets in $K_{3,3}$ are $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. We may describe our Hamiltonian circuit as starting and ending in vertex a. Since every element of S has an edge to every element of T, a Hamiltonian circuit is thus specified by giving a sequence alternating between letters (a,b,c) and numbers (1,2,3), such that no letter and no number is repeated (e.g. $a1b2c3$).

If C is a Hamiltonian circuit, then the order in which it passes through the elements of S is either a, b, c or a, c, b. Since Hamiltonian circuits are considered the same under reversing their direction, we can assume that we have chosen the direction so that the order is a, b, c. Thus all Hamiltonian circuits are of the form $axbycz$ where x, y, z denote the numbers $1, 2, 3$ in some order. Explicitly, we have the circuits

$$a1b2c3, a1b3c2, a2b3c1, a2b1c3, a3b1c2, a3b2c1.$$

c) Let S and T be the vertex sets of $K_{3,4}$, with $|S| = 3$ and $|T| = 4$. A Hamiltonian circuit in $K_{3,4}$ would have length 7. By a Theorem 6.4. of Aigner, a bipartite graph has no circuits of odd length, so this is impossible.

d) Let $\{1, \ldots, n\}$ be the vertex set of K_n. Starting from 1, a Hamiltonian circuit can be specified by listing the vertices it passes thrugh. This gives

word of length $n-1$ using all the numbers $2, \ldots, n$; conversely, every word of length $n-1$ using all the numbers $2, \ldots, n$ gives a Hamiltonian circuit.

As Hamiltonian circuits are considered the same under reversal of direction, a word $a_1 \ldots a_{n-1}$ and its reversed word $a_{n-1} \ldots a_1$ give the same circuit. Thus the number of Hamiltonian circuits is half the number of words. The number of words is $(n-1)!$, so the number of Hamiltonian circuits is $(n-1)!/2$.