UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in	MAT3360 — Introduction to Partial Differential Equations
Day of examination:	Thursday, June 7, 2018
Examination hours:	09:00 – 13:00

This problem set consists of 3 pages.

Appendices:	None.
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let f be the signum function, i.e.,

$$f(x) = \operatorname{sign}(x) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

1a

Find the full Fourier series for f, S_f, on the interval $[-1, 1]$.

1b

Sketch the graph of $S_f(x)$ for $x \in (-2, 2)$.

1c

Use the previous results to show that

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} \quad \text{and} \quad \frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

Problem 2

Consider the following partial differential equation

$$\begin{cases} u_t + q(x)u_x = u_{xx}, \quad x \in (0,1), \quad t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0, \\ u(x,0) = f(x). \end{cases} \tag{1}$$

Here q and f are continuous functions $[0,1] \to \mathbb{R}$,

(Continued on page 2.)

2a

Show the maximum principle

$$\min\left\{0, \min_{x\in(0,1)} f(x)\right\} \le u(x,t) \le \max\left\{0, \max_{x\in(0,1)} f(x)\right\}.$$

(**Hint**: Consider $v = u - \varepsilon t$ and let $\varepsilon \downarrow 0$). Explain why this implies that (1) can have at most one solution.

2b

Define

$$E(t) = \frac{1}{2}\int_0^1 (u(x,t))^2 \, dx.$$

Show that

$$E'(t) = -\int_0^1 (u_x)^2 \, dx - \frac{1}{2}\int_0^1 q(x)(u^2)_x \, dx. \tag{2}$$

Now we assume that q is continuously differentiable, such that $\|q'\|_\infty < \infty$, use (2) to establish the energy estimate

$$E(t) \le E(0)e^{\|q'\|_\infty t}.$$

2c

Let $h : [0,1] \to \mathbb{R}$ be a continuously differentiable function such that $h(0) = h(1) = 0$. Show the inequality

$$(h(x))^2 \le \min\{x, 1-x\} \int_0^1 \left(h'(y)\right)^2 \, dy \text{ for } x \in [0,1].$$

(**Hint**: Use that $|h(x)| = |\int_0^x h'(y)\,dy|$ and $|h(x)| = |\int_x^1 h'(y)\,dy|$ and the Cauchy-Schwartz inequality.)

2d

Show that $E(t) \le E(0)$ if

$$\int_0^1 \min\{x, 1-x\} \left|q'(x)\right| \, dx \le 2.$$

(**Hint**: Start with (2) and use **c**.)

Problem 3

Let $\Omega = \left\{(x,y) \mid x^2 + y^2 < 1\right\}$, and consider the following boundary value problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega, \\ u = g & (x,y) \in \partial\Omega, \end{cases} \tag{3}$$

where g is a given continuous function. In polar coordinates (r, φ) we have

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2},$$

(Continued on page 3.)

you do not have to show this. Let $\Delta r = 1/(N+1/2)$ and $\Delta\varphi = 2\pi/(M+1)$ for positive integers N and M, and set

$$r_i = (i-1/2)\Delta r,\ i = 1,\dots,N+1 \text{ and } \varphi_j = j\Delta\varphi,\ j = 0,\dots M+1.$$

We are interested in finding $u_{ij} \approx u(r_i, \varphi_j)$.

3a

Explain why the following difference scheme is a reasonable approximation to (3).

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta r} + \frac{1}{r_i^2}\frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{(\Delta\varphi)^2} = 0,$$
$$i = 1,\dots N,\quad j = 1,\dots,M,$$
$$u_{i,0} = u_{i,M},\ u_{i,1} = u_{i,M+1},\quad i = 1,\dots,N,$$
$$u_{N+1,j} = g(\varphi_j),\ j = 0,\dots,M.$$

3b

Let m_{ij} and M_{ij} be defined as the minimum and the maximum of $u_{\cdot,\cdot}$ at the neighboring points of (r_i, φ_j), i.e.,

$$m_{ij} = \begin{cases} \min\{u_{i+1,j}, u_{i-1,j}, u_{i,j+1}, u_{i,j-1}\} & N \geq i > 1 \\ \min\{u_{2,j}, u_{1,j+1}, u_{1,j-1}\} & i = 1, \end{cases}$$
$$M_{ij} = \begin{cases} \max\{u_{i+1,j}, u_{i-1,j}, u_{i,j+1}, u_{i,j-1}\} & N \geq i > 1 \\ \max\{u_{2,j}, u_{1,j+1}, u_{1,j-1}\} & i = 1. \end{cases}$$

Show the discrete maximum principle

$$m_{ij} \leq u_{ij} \leq M_{ij},\ i = 1,\dots,N,\ j = 1,\dots,M.$$

3c

Show that this implies that

$$\min_{\varphi\in[0,2\pi]} g(\varphi) \leq u_{ij} \leq \max_{\varphi\in[0,2\pi]} g(\varphi).$$

THE END