UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3360 — Introduction to Partial Differential Equations
Day of examination:	Tuesday 4 June 2019
Examination hours:	9:00 – 13:00
This problem set consists of 4 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: We recommend reading through the entire problem set before starting. The number of points given for each problem is stated in parentheses. The maximum number of points is 100.

Problem 1 (8 points)

Consider the following four problems:

$$\begin{cases} u_t(x,t) - 2(u_x(x,t))^2 = \sin x & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{A}$$

$$\begin{cases} u''(x) = 2e^x u'(x) & \text{for } x \in (-10,10) \\ u'(-10) = 0,\ u'(10) = 0 \end{cases} \tag{B}$$

$$\begin{cases} 3u_{xx}(x,t) - 2u_t(x,t) - x^2 = 0 & \text{for } x \in (0,1),\ t > 0 \\ u(0,t) = \cos(t),\ u(1,t) = \sin(t) & \text{for } t > 0 \\ u(x,0) = 0 \end{cases} \tag{C}$$

$$\begin{cases} u_t(x,y) - (u_x(x,t)k(x,t))_x = 0 & \text{for } x \in (0,1),\ t > 0 \\ u_x(0,t) = 0,\ u_x(1,t) = 0 & \text{for } t > 0 \\ u(x,t) = f(x). \end{cases} \tag{D}$$

For each of the above problems, specify

(i) whether it is an ODE or PDE

(ii) whether the equation is homogeneous or inhomogeneous

(iii) whether it is linear or nonlinear

(iv) the order of the equation (first, second, third, etc.)

(v) whether the boundary conditions (if any) are homogeneous or inhomogeneous, and if they are of Dirichlet or Neumann type.

(Continued on page 2.)

Problem 2 (30 points)

Consider the problem

$$\begin{cases} u_t = u_{xx} - \alpha u & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in [0,1] \end{cases} \tag{1}$$

where $\alpha > 0$ is a positive constant and $f : [0,1] \to \mathbb{R}$ is a given continuous function satisfying $f(0) = f(1) = 0$.

2a

Use separation of variables and the superposition principle to find a formal solution to the problem (1).

2b

Find the solution of (1) when $f(x) = 3\sin(2\pi x) - 5\sin(8\pi x)$.

2c

Explain how to find the solution of (1) when the boundary conditions have been replaced by $u(0,t) = u_0$, $u(1,t) = u_1$ for given constants $u_0, u_1 \in \mathbb{R}$. *(You do not have to compute the solution, only to explain the construction of the solution.)*

2d

Prove that any solution of (1) which is (at least) twice continuously differentiable satisfies the maximum principle

$$\min_{y\in[0,1]} f(y) \leqslant u(x,t) \leqslant \max_{y\in[0,1]} f(y) \qquad \forall\ x \in [0,1],\ t \geqslant 0. \tag{2}$$

2e

Use (2) to prove that (1) has at most one solution.

Problem 3 (5 points)

Consider the PDE

$$\begin{cases} u_{tt} = c^2 u_{xx} + k u_t & x \in (0,1),\ t > 0 \\ u(0,t) = u_0,\ u_x(1,t) = 0 & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & x \in (0,1) \end{cases} \tag{3}$$

for given numbers $u_0, c, k \in \mathbb{R}$ and continuous functions $f, g : [0,1] \to \mathbb{R}$. For what values of $k \in \mathbb{R}$ does the energy

$$E(t) := \int_0^1 \frac{u_t^2}{2} + \frac{c^2 u_x^2}{2}\,dx$$

decrease (or stay constant) over time? Justify your answer.

(Continued on page 3.)

Problem 4 (15 points)

4a

Consider the PDE

$$\begin{cases} u_{tt} = u_{xx} - \alpha u & x \in (0,1),\ t > 0 \\ u_x(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & x \in (0,1) \end{cases} \tag{4}$$

for a given number $\alpha > 0$ and continuous functions $f, g : [0,1] \to \mathbb{R}$. Find an "energy function" $E = E(t)$ depending on u such that $E(t) = E(0)$ for all $t > 0$.

4b

Use the energy function derived in the previous exercise to show that there exists at most one solution of (4).

Problem 5 (7 points)

Derive an explicit finite difference method for the problem

$$\begin{cases} u_t + (au)_x = g(u) & x \in (0,1),\ t > 0 \\ u(0,t) = u_0(t),\ u(1,t) = u_1(t) & t > 0 \\ u(x,0) = f(x) & x \in (0,1) \end{cases} \tag{5}$$

for a continuously differentiable function $a = a(x,t)$ and continuous functions u_0, u_1, f and g. *(You do not need to prove any properties of your numerical method.)*

Problem 6 (15 points)

Consider the transport equation on a periodic domain,

$$\begin{cases} u_t + cu_x = 0 & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) & t > 0 \\ u(x,0) = f(x) & x \in [0,1] \end{cases} \tag{6}$$

for some constant $c > 0$ and some continuous and bounded function $f : [0,1] \to \mathbb{R}$. We consider the implicit finite difference method

$$\begin{cases} \frac{v_j^{m+1} - v_j^m}{\Delta t} + c\frac{v_j^{m+1} - v_{j-1}^{m+1}}{\Delta x} = 0 & j = 1, \dots, n+1,\ m = 0, 1, \dots \\ v_0^{m+1} = v_{n+1}^{m+1} & m = 0, 1, \dots \\ v_j^0 = f(x_j) & j \in \mathbb{Z}. \end{cases} \tag{7}$$

Show that for any choice of $\Delta t, \Delta x > 0$, any solution of (7) satisfies

$$\inf_{x \in [0,1]} f(x) \leqslant v_j^m \leqslant \sup_{x \in [0,1]} f(x).$$

(Continued on page 4.)

Problem 7 (20 points)

Consider the heat equation

$$\begin{cases} u_t = u_{xx} & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in [0,1] \end{cases} \tag{8}$$

and consider the *leapfrog* finite difference method

$$\begin{cases} \frac{v_j^{m+1} - v_j^{m-1}}{2\Delta t} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2} & j = 1, 2, \ldots, n \\ v_0^m = v_{n+1}^m = 0 & m = 1, 2, \ldots \\ v_j^0 = f(x_j) & j = 1, 2, \ldots, n \\ v_j^1 = f(x_j) + \Delta t f''(x_j) & j = 1, 2, \ldots, n. \end{cases} \tag{9}$$

We assume that f is at least twice continuously differentiable in $[0,1]$.

7a

Explain the derivation of (9).

7b

Show that the finite difference method (9) is unconditionally unstable in the sense of von Neumann, that is, it is unstable for any choice of $\Delta t, \Delta x > 0$.

THE END