UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3360 — Introduction to Partial Differential Equations
Day of examination:	Tuesday 4 June 2019
Examination hours:	9:00 – 13:00

This problem set consists of 7 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: We recommend reading through the entire problem set before starting. The number of points given for each problem is stated in parentheses. The maximum number of points is 100.

Problem 1 (8 points)

Consider the following four problems:

$$\begin{cases} u_t(x,t) - 2(u_x(x,t))^2 = \sin x & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{A}$$

$$\begin{cases} u''(x) = 2e^x u'(x) & \text{for } x \in (-10,10) \\ u'(-10) = 0,\ u'(10) = 0 \end{cases} \tag{B}$$

$$\begin{cases} 3u_{xx}(x,t) - 2u_t(x,t) - x^2 = 0 & \text{for } x \in (0,1),\ t > 0 \\ u(0,t) = \cos(t),\ u(1,t) = \sin(t) & \text{for } t > 0 \\ u(x,0) = 0 \end{cases} \tag{C}$$

$$\begin{cases} u_t(x,y) - (u_x(x,t)k(x,t))_x = 0 & \text{for } x \in (0,1),\ t > 0 \\ u_x(0,t) = 0,\ u_x(1,t) = 0 & \text{for } t > 0 \\ u(x,t) = f(x). \end{cases} \tag{D}$$

For each of the above problems, specify

(i) whether it is an ODE or PDE

(ii) whether the equation is homogeneous or inhomogeneous

(iii) whether it is linear or nonlinear

(iv) the order of the equation (first, second, third, etc.)

(v) whether the boundary conditions (if any) are homogeneous or inhomogeneous, and if they are of Dirichlet or Neumann type.

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Solution:

(A) PDE, inhomogeneous, nonlinear, first-order

(B) ODE, homogeneous, linear, second-order, homogeneous Neumann boundary conditions

(C) PDE, inhomogeneous, linear, second-order, inhomogeneous Dirichlet boundary conditions

(D) PDE, homogeneous, linear, second-order, homogeneous Neumann boundary conditions.

Problem 2 (30 points)

Consider the problem

$$\begin{cases} u_t = u_{xx} - \alpha u & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in [0,1] \end{cases} \tag{1}$$

where $\alpha > 0$ is a positive constant and $f : [0,1] \to \mathbb{R}$ is a given continuous function satisfying $f(0) = f(1) = 0$.

2a

Use separation of variables and the superposition principle to find a formal solution to the problem (1).

Solution: The *ansatz* $u(x,t) = X(x)T(t)$ leads to $XT' = X''T - \alpha XT$, or $\frac{T'+\alpha T}{T} = \frac{X''}{X}$. Since the left-hand side is independent of x, and the right-hand side of t, both sides must be equal to some constant, say, $-\lambda \in \mathbb{R}$. This leads to the two ODEs

$$T' + \alpha T = -\lambda T, \qquad X'' = -\lambda X.$$

The equation for T has solutions $T(t) = ae^{-(\alpha+\lambda)t}$ for any $a \in \mathbb{R}$. The equation for X is supplemented with the boundary conditions $X(0) = X(1) = 0$, which always has the solutions $X \equiv 0$ and, when $\lambda = (k\pi)^2$ for some $k \in \mathbb{Z}$, $X(x) = \sin(k\pi x)$. Thus, the equation for X has particular solutions

$$\lambda_k = (k\pi)^2, \qquad X_k(x) = \sin(k\pi x), \qquad k \in \mathbb{N}.$$

In conclusion, (1) has the particular solutions

$$u_k(x,t) = a_k e^{-(\alpha+(k\pi)^2)t} \sin(k\pi x), \qquad k \in \mathbb{N}.$$

Letting $u(x,t) = \sum_{k\in\mathbb{N}} u_k(x,t)$ and asserting that $u(x,0) = f(x)$ shows

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that the coefficients a_k must be chosen as

$$a_k = \frac{\int_0^1 f(x)\sin(k\pi x)\,dx}{\int_0^1 \sin(k\pi x)^2\,dx} = 2\int_0^1 f(x)\sin(k\pi x)\,dx.$$

(No justification for the convergence of these Fourier series is required.)

2b

Find the solution of (1) when $f(x) = 3\sin(2\pi x) - 5\sin(8\pi x)$.

Solution: Here $a_2 = 3$, $a_8 = -5$ and $a_k = 0$ for $k \neq 2, 8$, so

$$u(x,t) = 3e^{-(\alpha+(2\pi)^2)t}\sin(2\pi x) - 5e^{-(\alpha+(8\pi)^2)t}\sin(8\pi x).$$

2c

Explain how to find the solution of (1) when the boundary conditions have been replaced by $u(0,t) = u_0$, $u(1,t) = u_1$ for given constants $u_0, u_1 \in \mathbb{R}$. *(You do not have to compute the solution, only to explain the construction of the solution.)*

Solution: First, find any solution v of the PDE satisfying the prescribed boundary conditions. For instance, the *ansatz* $v(x,t) = \phi(x)$ yields

$$\phi''(x) - \alpha\phi(x) = 0, \qquad \phi(0) = u_0,\ \phi(1) = u_1,$$

which has a solution of the form $\phi(x) = A\cos(\sqrt{\alpha}x) + B\sin(\sqrt{\alpha}x)$. Then, find the solution w of the homogeneous Dirichlet problem (1) but with initial data $w(x,0) = f(x) - v(x,0)$. The function $u = v + w$ now solves the problem in question.

2d

Prove that any solution of (1) which is (at least) twice continuously differentiable satisfies the maximum principle

$$\min_{y\in[0,1]} f(y) \leqslant u(x,t) \leqslant \max_{y\in[0,1]} f(y) \qquad \forall\, x \in [0,1],\ t \geqslant 0. \tag{2}$$

Solution: Let $T > 0$ and let $(x_0, t_0) \in [0,1]\times[0,T]$ be any point where

$$\max_{\substack{x\in[0,1]\\ t\in[0,T]}} u(x,t) = u(x_0,t_0).$$

Then either:

1. $x_0 = 0$ or $x_0 = 1$, in which case $u(x_0,t_0) = 0 \leqslant \max_{x\in[0,1]} f(x)$, the last inequality following from the fact that $f(0) = f(1) = 0$.
2. $t_0 = 0$, from which (2) directly follows.
3. $(x_0,t_0) \in (0,1)\times(0,T]$. Then $u_t(x_0,t_0) \geqslant 0$ and $u_{xx}(x_0,t_0) \leqslant 0$,

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so

$$0 \leqslant u_t(x_0,t_0) - u_{xx}(x_0,t_0) = -\alpha u(x_0,t_0).$$

Since $\alpha > 0$, this implies that $u(x_0,t_0) \leqslant 0 \leqslant \max_{x\in[0,1]} f(x)$, the last inequality again following from the fact that $f(0) = f(1) = 0$.

The lower bound in (2) follows from replacing u and f by $-u$ and $-f$, respectively.

2e

Use (2) to prove that (1) has at most one solution.

Solution: Let u and v be two solutions with the same initial data. Then by linearity $w := u - v$ solves (1) with $w(x,0) = 0$. From (2) it follows that

$$0 \leqslant w(x,t) \leqslant 0 \qquad \forall\, x \in [0,1],\ t \geqslant 0,$$

whence $u = v$.

Problem 3 (5 points)

Consider the PDE

$$\begin{cases} u_{tt} = c^2 u_{xx} + k u_t & x \in (0,1),\ t > 0 \\ u(0,t) = u_0,\ u_x(1,t) = 0 & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & x \in (0,1) \end{cases} \tag{3}$$

for given numbers $u_0, c, k \in \mathbb{R}$ and continuous functions $f, g : [0,1] \to \mathbb{R}$. For what values of $k \in \mathbb{R}$ does the energy

$$E(t) := \int_0^1 \frac{u_t^2}{2} + \frac{c^2 u_x^2}{2}\,dx$$

decrease (or stay constant) over time? Justify your answer.

Solution: We have

$$E'(t) = \int_0^1 u_t u_{tt} + c^2 u_x u_{xt}\,dx = \int_0^1 u_t(u_{tt} - c^2 u_{xx})\,dx + c^2 [u_x u_t]_{x=0}^{x=1}.$$

Since $u(0,t)$ is constant we have $u_t(0,t) = 0$, and moreover $u_x(1,t) = 0$. Hence,

$$E'(t) = \int_0^1 u_t(u_{tt} - c^2 u_{xx})\,dx = k \int_0^1 u_t(x,t)^2\,dx.$$

It follows that $E'(t) \leqslant 0$ for all t *if and only if* either

1. u is constant in time, i.e. $u_t \equiv 0$, or
2. $k \leqslant 0$.

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Problem 4 (15 points)

4a

Consider the PDE

$$\begin{cases} u_{tt} = u_{xx} - \alpha u & x \in (0,1),\ t > 0 \\ u_x(0,t) = u_x(1,t) = 0 & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & x \in (0,1) \end{cases} \tag{4}$$

for a given number $\alpha > 0$ and continuous functions $f, g : [0,1] \to \mathbb{R}$. Find an "energy function" $E = E(t)$ depending on u such that $E(t) = E(0)$ for all $t > 0$.

Solution: The function

$$E(t) = \int_0^1 \frac{u_t(x,t)^2}{2} + \frac{u_x(x,t)^2}{2} + \alpha \frac{u(x,t)^2}{2}\, dx$$

(or $\gamma E(t)$ for any $\gamma > 0$) is an energy for (4).

4b

Use the energy function derived in the previous exercise to show that there exists at most one solution of (4).

Solution: If u and v are solutions of (4), let $w = u - v$. Then w solves the same problem but with zero initial data. Hence, the energy $E(t)$ of w is equal to its initial energy $E(0) = 0$. But $E(t) = 0$ is equivalent to $w(x,t) = 0$ for all x. Hence, $u = v$.

Problem 5 (7 points)

Derive an explicit finite difference method for the problem

$$\begin{cases} u_t + (au)_x = g(u) & x \in (0,1),\ t > 0 \\ u(0,t) = u_0(t),\ u(1,t) = u_1(t) & t > 0 \\ u(x,0) = f(x) & x \in (0,1) \end{cases} \tag{5}$$

for a continuously differentiable function $a = a(x,t)$ and continuous functions u_0, u_1, f and g. *(You do not need to prove any properties of your numerical method.)*

Solution: Let $n \in \mathbb{N}$, $\Delta x = \frac{1}{n+1}$, $x_j = j\Delta x$ for $j = 0, \dots, n+1$ and $t_m = m\Delta t$ for some $\Delta t > 0$. Letting $v_j^m \approx u(x_j, t_m)$ and approximating the temporal and spatial derivatives in (5) by e.g. forward and central differences, respectively, yields

$$\begin{cases} \frac{v_j^{m+1} - v_j^m}{\Delta t} + \frac{a_{j+1}^m v_{j+1}^m - a_{j-1}^m v_{j-1}^m}{2\Delta x} = g(v_j^m) & j = 1, \dots, n,\ m = 0, 1, \dots \\ v_0^m = u_0(t_m),\ v_{j+1}^m = u_1(t_m) & m = 1, 2, \dots \\ v_j^0 = f(x_j) & j = 1, \dots, n \end{cases} \tag{6}$$

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where $a_j^m = a(x_j, t_m)$.

Problem 6 (15 points)

Consider the transport equation on a periodic domain,

$$\begin{cases} u_t + cu_x = 0 & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) & t > 0 \\ u(x,0) = f(x) & x \in [0,1] \end{cases} \tag{7}$$

for some constant $c > 0$ and some continuous and bounded function $f : [0,1] \to \mathbb{R}$. We consider the implicit finite difference method

$$\begin{cases} \frac{v_j^{m+1} - v_j^m}{\Delta t} + c\frac{v_j^{m+1} - v_{j-1}^{m+1}}{\Delta x} = 0 & j = 1, \dots, n+1,\ m = 0, 1, \dots \\ v_0^{m+1} = v_{n+1}^{m+1} & m = 0, 1, \dots \\ v_j^0 = f(x_j) & j \in \mathbb{Z}. \end{cases} \tag{8}$$

Show that for any choice of $\Delta t, \Delta x > 0$, any solution of (8) satisfies

$$\inf_{x\in[0,1]} f(x) \leqslant v_j^m \leqslant \sup_{x\in[0,1]} f(x).$$

Solution: Let J be a point at which $(v_j^{m+1})_{j=1}^{n+1}$ attains its maximum. Then

$$v_J^{m+1} = v_J^m - c\frac{\Delta t}{\Delta x}(v_J^{m+1} - v_{J-1}^{m+1}).$$

By assumption, $c > 0$, and by the choice of J we have $v_J^{m+1} - v_{J-1}^{m+1} \geqslant 0$. It follows that

$$\max_{j=0,\dots,n+1} v_j^{m+1} = v_J^{m+1} \leqslant v_J^m \leqslant \max_{j=0,\dots,n+1} v_j^m.$$

Iterating the inequality over all m yields

$$v_j^m \leqslant \max_{j=0,\dots,n+1} v_j^0 \leqslant \max_{x\in[0,1]} f(x).$$

The lower bound is shown in the same way.

Problem 7 (20 points)

Consider the heat equation

$$\begin{cases} u_t = u_{xx} & x \in (0,1),\ t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = f(x) & x \in [0,1] \end{cases} \tag{9}$$

and consider the *leapfrog* finite difference method

$$\begin{cases} \frac{v_j^{m+1} - v_j^{m-1}}{2\Delta t} = \frac{v_{j-1}^m - 2v_j^m + v_{j+1}^m}{\Delta x^2} & j = 1, 2, \dots, n \\ v_0^m = v_{n+1}^m = 0 & m = 1, 2, \dots \\ v_j^0 = f(x_j) & j = 1, 2, \dots, n \\ v_j^1 = f(x_j) + \Delta t f''(x_j) & j = 1, 2, \dots, n. \end{cases} \tag{10}$$

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We assume that f is at least twice continuously differentiable in $[0, 1]$.

7a

Explain the derivation of (10).

Solution: The time derivative is approximated by a central difference and the spatial derivative by a central, second-order difference. The data for v_0^m, v_{n+1}^m and v_j^m are obvious, while

$$v_j^1 \approx u(x_j, \Delta t) \approx u(x_j, 0) + \Delta t u_t(x_j, 0) = f(x_j) + \Delta t u_{xx}(x_j, 0)$$
$$= f(x_j) + \Delta t f''(x_j).$$

7b

Show that the finite difference method (10) is unconditionally unstable in the sense of von Neumann, that is, it is unstable for any choice of $\Delta t, \Delta x > 0$.

Solution: We know that the heat equation has particular solutions $u_k(x, t) = e^{-(k\pi)^2 t} e^{ik\pi x}$, and that these satisfy $|u_k(x, t)| \leqslant 1$. Hence, it makes sense to make the *ansatz* $v_j^m = a^m e^{ik\pi x_j}$ for some $a \in \mathbb{C}$ and $k \in \mathbb{Z}$, and require that $|v_j^m| \leqslant 1$, i.e. $|a| \leqslant 1$. Inserting into the difference equation (10) yields

$$e^{ik\pi x_j} \frac{a^{m+1} - a^{m-1}}{2\Delta t} = a^m e^{ik\pi x_j} \frac{e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}}{\Delta x^2}.$$

After simplifying,

$$\frac{a^2 - 1}{2\Delta t} = a \frac{e^{-ik\pi\Delta x} - 2 + e^{ik\pi\Delta x}}{\Delta x^2} = -a\mu_k$$

where $\mu_k = 4\frac{\sin(k\pi\Delta x/2)^2}{\Delta x^2}$ is the kth eigenvalue of the discrete Laplacian. This yields

$$a = \frac{-2\mu_k\Delta t \pm \sqrt{4(\mu_k\Delta t)^2 + 4}}{2} = -\mu_k\Delta t \pm \sqrt{(\mu_k\Delta t)^2 + 1}.$$

Both of these roots are real, and the smallest root is strictly smaller than -1, regardless of the values of $\Delta t, \Delta x$. Hence, no CFL condition can ensure that the amplification factor a is smaller than 1 in magnitude.

THE END