UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Examination in:	INF-MAT3360 — Partial Differential Equations.
Day of examination:	26. March 2007.
Examination hours:	13.30 – 16.30.

This examination set consists of 3 pages.

Appendices:	Ingen
Permitted aids:	Ingen.

Make sure that your copy of the examination set is complete before you start solving the problems.

Problem 1.

Consider the following ODE,

$$\begin{cases} u'(t) = -tu(t), & t \geq 0, \\ u(0) = u_0 \end{cases} \tag{1}$$

(i.) Find an explicit solution of the ODE (1).

Let $u_\epsilon(t)$ be the solution of the following ODE,

$$\begin{cases} u'_\epsilon(t) = -(t+\epsilon^2)u_\epsilon(t), & t \geq 0, \\ u_\epsilon(0) = (u_0 + \epsilon^{987}) \end{cases} \tag{2}$$

(ii.) Show that for any $t \in [0, \infty)$, we have

$$\lim_{\epsilon \to 0} u_\epsilon(t) = u(t)$$

where u is the solution of (1).

(iii.) Write down a finite difference scheme of the Forward Euler type to numerically solve the ODE (1).

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Consider the following ODE,

$$\begin{cases} u'(t) = -tu^2(t), & t \geq 0, \\ u(0) = u_0 \end{cases} \tag{3}$$

(iv.) Write the down the explicit solution of ODE (3) when $u_0 = 1$.

(v.) If $u_0 = -1$ in (3). Then find $u(\sqrt{2})$ and $u(2)$ where u is the correspon-ding solution of (3).

Problem 2.

Consider the following first order PDE,

$$\begin{cases} u_t + au_x = x, & x \in \mathbb{R}, t \in [0, \infty), \\ u(x, 0) = \phi(x), & x \in \mathbb{R} \end{cases} \tag{4}$$

where $a \in \mathbb{R}$.

(i.) Find an explicit solution u of (4) by using the method of characteristics.

(ii.) If u and v are two solutions of the PDE (4), show that $u \equiv v$ i.e $u(x, t) = v(x, t)$ for all $x \in \mathbb{R}$ and $t \in [0, \infty)$.

Consider the following PDE,

$$\begin{cases} u_t + 2u_x - xu_x = x^2 + t^2, & x \in \mathbb{R}, t \in [0, \infty), \\ u(x, 0) = \phi(x), & x \in \mathbb{R} \end{cases} \tag{5}$$

(iii.) Find an explicit solution u of (5).

Problem 3.

Consider the Poisson's equation in one dimension given by,

$$\begin{cases} -u''(x) = f(x), & x \in (0, 1). \\ u(0) = u(1) = 0. \end{cases} \tag{6}$$

The PDE (6) can be rewritten in the following operator form,

$$Lu = f \tag{7}$$

where the linear Differential operator L is defined as

$$L : C_0^2((0, 1)) \mapsto C^0((0, 1)) \quad \text{with} \quad Lu = -u''$$

then,

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(i.) Show that L is a symmetric and positive definite operator.

(ii.) If u and v are two solutions of (7), then show that $u \equiv v$.

Consider the following two-point boundary value problem,

$$\begin{cases} -u''(x) + \lambda u(x) = f(x), & x \in (0,1). \\ u(0) = u(1) = 0. \end{cases} \tag{8}$$

where $\lambda > 0$. The PDE (8) can be re-written in the operator form as

$$\overline{L}u = f \tag{9}$$

where the linear Differential operator L is defined as

$$\overline{L} : C_0^2((0,1)) \mapsto C^0((0,1)) \quad \text{with} \quad \overline{L}u = -u'' + \lambda u$$

then,

(iii.) Show that $\overline{L}$ defined above is symmetric and positive definite.

(iv.) Write down a finite difference scheme for numerically computing the solution of (8). Check that the discrete solution will be a solution for a linear system of the form,

$$Av(x_j) = f(x_j) \tag{10}$$

where $x_j = jh$, $h = 1/n$ and $n > 0$ is a uniform discretization of the interval $[0,1]$ with n points, v is the discrete solution vector and A is a matrix. Write down A explicitly.

(v.) Prove that Gaussian elimination is well defined for the linear system (10) obtained above. (HINT: Check the structure of Matrix A).

(vi.) If $f = 0$ and $\lambda = 5012\pi$, then what is the solution of equation (8). Is it unique ? .

(vii.) if $f = 0$ and $\lambda = 1000$, then find the solution u of (8).

END