SOLUTION

Problem 1

1-a

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x) - 4e^{-9\pi^2 t}\sin(3\pi x)$$

This is an analog of Example 3.1 in the book.

1-b The sine Fourier series of is first computed as

$$x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin(k\pi x)$$

and hence the solution is given by

$$x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}e^{-(k\pi)^2 t}\sin(k\pi x).$$

This is also an example from the book (Example 3.4).

1-c Consider $v(x,t) = u(x,t) - 2x$. Then v has homogeneous boundary conditions and satisfies the heat equation. Furthermore, $v(x,0) = \sin(\pi x) - 4\sin(3\pi x) - 2x$ and from above we have

$$v(x,t) = e^{-\pi^2 t}\sin(\pi x) - 4e^{-9\pi^2 t}\sin(3\pi x) - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}e^{-(k\pi)^2 t}\sin(k\pi x).$$

Finally, $u(x,t) = v(x,t) + 2x$. Here we have used the technique from Exercise 3.11 in the book.

Problem 2

2-a The characteristics are of the form $x(t) = x_0 + at$ and the solution u is constant on the characteristics. Therefore,

$$u(x,t) = \begin{cases} f(x-at) & \text{if } t < x/a \\ 0 & t \geq x/a \end{cases}$$

Since u only takes values f or zero the estimate is obvious and u is identical zero for $t > 1/a$. (see Section 1.4 in the book.)

2-b The explicit scheme can be rewritten as

$$v_j^{m+1} = (1-s)v_j^m + sv_{j-1}^m,$$

where $s = a\frac{\Delta t}{\Delta x}$. The desired estimate will hold if $1 - s \geq 0$, i.e. if $a\frac{\Delta t}{\Delta x} \leq 1$.

2-c The implicit scheme can be rewritten as

$$v_j^{m+1} = (1+s)^{-1} v_j^m + s(1+s)^{-1} v_{j-1}^{m+1}$$

Since the coefficients $(1+s)^{-1}$ and $s(1+s)^{-1}$ are always positive, and sum up to one the estimate follows by induction with respect to m and j. If we compute v_j^{m+1} in the order $v_1^{m+1}, v_2^{m+1}, v_n^{m+1}$ there is no need to solve a linear system.

Problem 3

3-a The identity follows by multiplying the differential equation by u and integration by parts. If $f = 0$ then this identity implies that $u = 0$, so we have uniqueness.

3-b Constants satisfies the boundary value problem with $f = 0$ so we do not have uniqueness. Furthermore, if u is a solution then

$$\int_0^1 f(x)\,dx = -\int_0^1 u_{xx}(x)\,dx = 0$$

by the boundary condition.

3-c Assume $f = 0$ and $a = 0$. Then we will have

$$\int_0^1 (u_x(x,t))^2\,dx + \lambda \int_0^1 u(x)\,dx = \int_0^1 (u_x(x,t))^2\,dx = 0,$$

so u is a constant, and it has integral equal to zero. So $u = 0$, and then $\lambda = 0$ by (12).

3-d It follows from 3-b above that

$$\lambda = \int_0^1 f(x)\,dx = c_0/2.$$

Furthermore, by writing u as a Fourier cosine series on the form

$$u(x) = \frac{b_0}{2} + \sum_{k=1}^{\infty} b_k \cos(k\pi x)$$

we have that

$$b_0 = 2\int_0^1 u(x)\,dx = 2a.$$

Finally, by inserting the expansion of u and f into the equation (12) we obtain $b_k = c_k/(k\pi)^2$. So the solution is

$$u(x) = a + \sum_{k=1}^{\infty} \frac{c_k}{(k\pi)^2} \cos(k\pi x), \qquad \text{and} \quad \lambda = c_0/2.$$

(see Section 3.6 in the book for the cosine series.)