UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF 3360 — Introduction to partial differential equations

Day of examination: June 8, 2015

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1-a

Solve the following problem related to the heat equation

$$u_t = u_{xx}, \quad x \in (0,1),\, t > 0, \tag{1}$$
$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{2}$$
$$u(x,0) = \sum_{k=1}^{100} c_k \sin(k\pi x), \quad x \in (0,1), \tag{3}$$

where $c_1, c_2, \ldots, c_{100}$ are given real coefficients.

1-b

Solve the problem (1), with boundary condition (2), and with initial condition given by

$$u(x,0) = x(1-x), \quad x \in (0,1). \tag{4}$$

1-c

Use separation of variables to solve the equation

$$u_t - u_{xxt} = u_{xx}, \quad x \in (0,1),\, t > 0,$$

with boundary conditions (2), and with initial condition (3).

(Continued on page 2.)

Problem 2

2-a

Consider the "backward heat equation" given by

$$u_t = -u_{xx}, \quad x \in (0,1),\, t > 0, \tag{5}$$
$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{6}$$
$$u(x,0) = f(x), \quad x \in (0,1), \tag{7}$$

where you should notice the minus sign in (5). Determine a solution in the case when $f(x) = \sin(k\pi x)$, where $k > 0$ is an integer.

2-b

Let $t > 0$ be fixed. Explain why there is no constant $C > 0$, independent of initial function f, such that

$$\int_0^1 u^2(x,t)\, dx \leq C \int_0^1 f^2(x)\, dx.$$

Is the problem (5) – (7) stable with respect to perturbations of the initial function? Justify your answer.

2-c

Assume that we replace (5) by the modified equation

$$u_t - \delta u_{xxt} = -u_{xx}, \quad x \in (0,1),\, t > 0, \tag{8}$$

where $\delta > 0$ is a small parameter. Show that smooth solutions of the problem (6) – (8) will satisfy

$$E(t) \leq e^{2t/\delta} E(0), \quad \text{where } E(t) = \int_0^1 [u^2(x,t) + \delta u_x^2(x,t)]\, dx.$$

Problem 3

3-a

We consider the boundary value problem

$$\epsilon^2 u''(x) + 2xu'(x) = 0, \quad x \in (0,1), \tag{9}$$
$$u(0) = a,\, u(1) = b, \tag{10}$$

where a, b are real numbers and $\epsilon > 0$. Show that the function

$$u_\epsilon(x) = a + \frac{(b-a)H(x/\epsilon)}{H(1/\epsilon)},$$

where $H(z) = \int_0^z e^{-t^2}\, dt$, solves this problem and use this to show that

$$\min(a,b) \leq u_\epsilon(x) \leq \max(a,b), \quad x \in [0,1]. \tag{11}$$

(Continued on page 3.)

3-b

For all $x \in [0, 1]$ determine the limit of $u_\epsilon(x)$ as $\epsilon \to 0$. Is the convergence uniform on $[0, 1]$? Sketch the function u_ϵ when ϵ is small, $a = 1$, and $b = 2$.

3-c

Assume that the real sequence $\{v_j\}_{j=0}^n$ satisfies the difference equation

$$v_j = \alpha_j v_{j+1} + \beta_j v_{j-1}, \quad j = 1, 2, \ldots, n,$$

where the coefficients satisfy

$$\alpha_j, \beta_j > 0, \ \alpha_j + \beta_j = 1, \quad j = 1, 2, \ldots, n.$$

Show that

$$\min(v_0, v_{n+1}) \leq v_j \leq \max(v_0, v_{n+1}), \quad j = 1, 2, \ldots, n.$$

3-d

Assume that the problem (9), (10) is approximated by the difference scheme

$$\epsilon^2 \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + 2x_j \frac{v_{j+1} - v_{j-1}}{2h} = 0, \quad 1 \leq j \leq n, \tag{12}$$

$$v_0 = a, \quad v_{n+1} = b, \tag{13}$$

where $x_j = jh$. Show that if $h < \epsilon^2$ then every solution of the difference scheme will satisfy the following discrete analog of (11):

$$\min(a, b) \leq v_j \leq \max(a, b) \quad 0 \leq j \leq n + 1. \tag{14}$$

3-e

Show by a counterexample that if ϵ is small compared to h then the discrete maximum principle (14) will not hold. (Hint: Consider the case $n = 1$.)