SOLUTION

Problem 1

1-a

$$u(x,t)=\sum_{k=1}^{100} c_k e^{-(k\pi)^2 t}\sin(k\pi x)$$

This follows from Chapter 3 of the book.

1-b The sine Fourier series of $x(1-x)$ is first computed as

$$x(1-x)=\sum_{k=1}^{\infty} c_k \sin(k\pi x),\quad \text{where } c_k=\frac{4}{(k\pi)^3}[1-(-1)^k]=\begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{8}{(k\pi)^3} & \text{if } k \text{ is odd}\end{cases}$$

As a consequence, the solution is given by

$$u(x,t)=\sum_{k=1}^{\infty} c_k e^{-(k\pi)^2}\sin(k\pi x)$$

This follows from Chapter 3 of the book.

1-c We make the ansatz that $u(x,t)=X(x)T(t)$. Inserting this into the equation gives

$$\frac{T'}{T}=\frac{X''}{X}(1-\frac{X''}{X})=\lambda,$$

where λ is independent of x and t. Furthermore, $\mu=\lambda/(1+\lambda)$ is an eigenvalue of the standard eigenvalue problem

$$-X''=\mu X,\quad X(0)=X(1)=0,$$

where the boundary conditions is a consequence of (2). So by standard theory we have $X(x)=X_k(x)=\sin(k\pi x)$ and $\mu=\mu_k=-(k\pi)^2$. This gives $\lambda=\lambda_k=-(k\pi)^2/(1+(k\pi)^2)$, and particular solutions

$$u_k(x,t)=e^{\lambda_k t}\sin(k\pi x).$$

For initial data of the form (3) this gives

$$u(x,t)=\sum_{k=1}^{100} c_k e^{\lambda_k t}\sin(k\pi x).$$

Problem 2

2-a The ansatz $u(x,t) = T(t)\sin(k\pi x)$ gives
$$u(x,t) = e^{(k\pi)^2 t}\sin(k\pi x).$$

2-b From the special solutions above we see that
$$C \geq e^{2(k\pi)^2 t} \to \infty \quad k \to \infty.$$
So no finite C exists, and the problem is unstable, since small perturbations of the form $\epsilon \sin(k\pi x)$ of the initial function can lead to arbitrary large changes of the solution.

2-c By using the differential equation, the boundary conditions and integration by parts we obtain
$$\begin{aligned} E'(t) &= 2\int_0^1 [uu_t + u_x u_{xt}]\,dx = 2\int_0^1 u(u_t - u_{xxt})\,dx \\ &= -2\int_0^1 uu_{xx}\,dx = 2\int_0^1 u_x^2\,dx \\ &\leq (2/\delta)E(t). \end{aligned}$$
The desired inequality follows from Gronwall.

Problem 3

3-a The boundary conditions are satisfied since $H(0) = 0$. Furthermore, for $u = u_\epsilon$ we have
$$u' = (C/\epsilon)e^{-(x/\epsilon)^2},$$
where $C = (b-a)/H(1/\epsilon)$. The chain rule now gives
$$u'' = -\frac{2xC}{\epsilon^3}e^{-(x/\epsilon)^2} = -\frac{2x}{\epsilon^2}u'.$$
As a consequence the differential equation holds. Finally, since H is increasing the maximum principle holds.

3-b We observe that for any positive x we have
$$\lim_{\epsilon\to 0} H(x/\epsilon) = H(\infty) < \infty.$$
As a consequence
$$\lim_{\epsilon\to 0} u_\epsilon(x) = a + \frac{(b-a)H(\infty)}{H(\infty)} = b$$
for $x > 0$, while $u_\epsilon(0) = a$ for all ϵ. So the limit is discontinuous and therefore no uniform convergence, except for the special case when $a = b$.

3-c We obtain that
$$\min(v_{j-1}, v_{j+1}) \leq v_j \leq \max(v_{j-1}, v_{j+1}).$$
This leads to the maximum principle by assuming that it does not hold.

3-d Reformulate the scheme and use point c). In fact, we obtain a difference equation of the form above with $\alpha_j = \frac{1}{2}(1 + x_j h/\epsilon^2)$ and $\beta_j = \frac{1}{2}(1 - x_j h/\epsilon^2)$. If $h < \epsilon^2$ the conditions given above are easily checked.

3-e Take $n = 1$. Then $h = 1/2$ and

$$v_1 = \frac{1}{2}(1 + 1/(4\epsilon^2))b + \frac{1}{2}(1 - 1/(4\epsilon^2))a.$$

If $a = 0$ and $b = 1$ then $v_1 = \frac{1}{2}(1 + 1/(4\epsilon^2)$. So for $\epsilon = 1/3$ we have $v_1 = 13/8 > 1$. This shows that (14) does not hold.