UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT-INF 3360 — Introduction to partial differential equations
Day of examination:	June 13, 2016
Examination hours:	14.30 – 18.30

This problem set consists of 5 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Solution.

Problem 1

Let $a \geq 0$ and consider the problem

$$\begin{aligned} u_t &= u_{xx}, && x \in (0,1),\ t > 0, \\ u(0,t) &= u(1,t) = a, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned} \tag{1}$$

1a

Solve equation (1) when $a = 0$ and $f(x) = \sin(2\pi x) - \sin(5\pi x)$. (You may use a general formula for the solution, or make the computations.)

Solution: From Chapter 3 in the book, we know that the solution is

$$u(x,t) = e^{-(2\pi)^2 t} \sin(2\pi x) - e^{-(5\pi)^2 t} \sin(5\pi x).$$

□

1b

Find the formal solution of equation (1) when $f(x) = 1$ and $a \geq 0$.

Solution: Introduce the function $v(x,t) = u(x,t) - a$. Then u is a solution of the desired problem if and only if v solves $v_t = v_{xx}$, $v(0,t) = v(1,t) = 0$ and $v(x,0) = 1 - a$ (for all x, t). From the book the solution v is then

$$v(x,t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t} \sin(k\pi x)$$

(Continued on page 2.)

where $c_k = 2\int_0^1 (1-a)\sin(k\pi x)dx = (1-a)\cdot 2\int_0^1 \sin(k\pi x)dx$ which gives $c_k = \frac{4(1-a)}{k\pi}$ for $k = 1, 3, 5, \ldots$ and $c_k = 0$ for $k = 2, 4, 6, \ldots$. This specifies v and the desired solution is then $u(x,t) = a + v(x,t)$.

$\square$

Problem 2

Consider the initial-boundary value problem of the form

$$\begin{cases} u_t + au_x = 0, & x \in [0,1],\ t > 0, \\ u(0,t) = t, & t > 0, \\ u(x,0) = f(x), & x \in [0,1] \end{cases} \tag{2}$$

where $a > 0$ is a constant, and f is a given continuously differentiable function.

2a

Use the method of characteristics to show that any solution of (2) satisfies

$$u(x,t) = \begin{cases} f(x-at) & \text{for} \quad x > at, \\ t - x/a & \text{for} \quad x < at \end{cases} \tag{3}$$

and where we assume $x \in [0,1]$ and $t \geq 0$.

Solution: The equation of the characteristic is $x'(t) = a, x(0) = x_0$ with solution $x(t) = x_0 + at$. So the solution u is constant on straight lines $L_{x_0} = \{(x,t) : x = x_0 + at\}$. Let $x \in [0,1]$ and $t \geq 0$. If $x > at$, then $(x,t) \in L_{x_0}$ with $x_0 = x - at > 0$, so $u(x,t) = u(x_0, 0) = f(x_0) = f(x-at)$. If $x < at$, then $(x,t) \in L_{x_0}$ with $x_0 = x - at < 0$, so $u(x,t) = u(0, t - x/a) = t - x/a$. $\square$

Let $u(x,t)$ be as in (3), and define

$$u(x,t) = 0 \quad \text{when } x = at,\ x \in [0,1]. \tag{4}$$

2b

Give conditions on f at $x = 0$ such that u, given by (3) and (4), is continuous. Moreover, prove that u satisfies

$$|u(x,t)| \leq \max\{t, M\} \quad x \in [0,1],\ t > 0. \tag{5}$$

where $M = \max\{|f(x_1)| : x_1 \in [0,1]\}$.

Solution: Let (x_1, t_1) satisfy $x_1 = at_1$ (and $x_1 \in [0,1]$ and $t_1 \geq 0$). For u to be continuous at (x_1, t_1) we must have $\lim_{(x,t)\to(x_1,t_1)} u(x,t) = u(x_1, t_1) = 0$, so therefore $\lim_{(x,t)\to(x_1,t_1)} f(x - at) = 0$, and since f is continuous, this means that $f(0) = 0$. This condition also implies that u is continuous.

To show (5): Let $x \in [0,1]$ and $t \geq 0$. If $x = at$, then $u(x,t) = 0$. If $x > at$, then $|u(x,t)| = |f(x-at)| \leq M$. Finally, if $x < at$, then $u(x,t) = t - x/a$, so $0 < u(x,t) < t$ and $|u(x,t)| < t$. This gives the bound in (5). $\square$

(Continued on page 3.)

Problem 3

Let $n \geq 1$, $h = 1/(n+1)$ and define grid points $(x_j, y_k) = (jh, kh)$ for $0 \leq j, k \leq n+1$. Let v be a grid function, with $v_{j,k} = v(x_j, y_k)$. Recall that v is called a *discrete harmonic function* if

$$L_h v(x_j, y_k) = 0 \qquad 1 \leq j, k \leq n$$

where the finite difference operator L_h is defined by

$$(L_h v)(x_j, y_k) = \frac{1}{h^2}[4v_{j,k} - v_{j+1,k} - v_{j-1,k} - v_{j,k+1} - v_{j,k-1}].$$

3a

Assume that v is a discrete harmonic function. Show that, for each $1 \leq j, k \leq n$,

$$m_{jk} \leq v_{j,k} \leq M_{jk} \tag{6}$$

where

$$\begin{aligned} m_{jk} &= \min\{v_{j+1,k}, v_{j-1,k}, v_{j,k+1}, v_{j,k-1}\}, \\ M_{jk} &= \max\{v_{j+1,k}, v_{j-1,k}, v_{j,k+1}, v_{j,k-1}\}. \end{aligned}$$

Solution: From $L_h v(x_j, y_k) = 0$ we get $v_{j,k} = (1/4)(v_{j+1,k} + v_{j-1,k} + v_{j,k+1} + v_{j,k-1})$. Therefore

$$v_{j,k} \geq (1/4)(4m_{jk}) = m_{jk} \quad \text{and} \quad v_{j,k} \leq (1/4)(4M_{jk}) = M_{jk}.$$

□

3b

Use (6) to establish a maximum principle for discrete harmonic functions, i.e., that the maximum, and the minimum, of such a function v is attained at a grid point on the boundary.

Solution: Let $I = \{(j,k) : v_{j,k} = M_{jk}\}$. Assume that $(j,k) \in I$ for an interior (j,k), i.e., with $1 \leq j, k \leq n$. Since $v_{j,k}$ is the mean (average) of its four neighbor values, all these neighbors must have the value $M_{j,k}$, too. Repeating this, for the neighbors, we eventually reach the boundary (when j or k is 0 or $n+1$) and the value there is also $M_{j,k}$. So, the maximum of v is attained at the boundary. Similarly one shows that the minimum of v is attained at the boundary. □

Problem 4

Consider the initial and boundary value problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) - q(x)u(x,t), && x \in (0,1),\ t > 0, \\ u(0,t) &= u(1,t) = 0, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned} \tag{7}$$

Here q is a given continuous function which satisfies $q(x) \geq 0$ for all $x \in [0,1]$. Let $u = u(x,t)$ be a solution of (7), and define, for each $t \geq 0$, the energy

$$E(t) = \int_0^1 u^2(x,t)dx.$$

(Continued on page 4.)

4a

Use energy arguments to show that

$$E(t) \leq \int_0^1 f^2(x)dx \quad \text{for } t > 0. \tag{8}$$

Hint: You may assume that it is possible to interchange the order of differentiation and integration.

Solution: We use the hint, insert from PDE and use integration by parts and obtain for $t > 0$

$$\begin{aligned} E'(t) &= \tfrac{d}{dt}\int_0^1 u^2(x,t)dx \\ &= \int_0^1 \tfrac{\partial}{\partial t}u^2(x,t)dx \\ &= \int_0^1 2uu_t dx \\ &= \int_0^1 2u(u_{xx} - qu)dx \\ &= 2([uu_x]_0^1 - \int_0^1 u_x^2 dx - \int_0^1 qu^2 dx) \\ &= (-2)(\int_0^1 u_x^2 dx + \int_0^1 qu^2 dx) \\ &\leq 0 \end{aligned}$$

We here used that $u(0) = u(1) = 0$, and that $q \geq 0$. This proves that $E'(t) \leq 0$ for all $t > 0$. So E is nonincreasing, and the desired inequality follows as $E(0) = \int_0^1 f^2(x)dx$. □

4b

Use the inequality in (8) to show that the PDE in (7) has at most one solution.

Solution: Assume that there are two solutions u^1 and u^2, and let $w = u^1 - u^2$. Then, by linearity, w satisfies (7) with f replaced by the zero function. But then we get from (8) that

$$\int_0^1 w^2 dx = \int_0^1 (u^1 - u^2)^2 dx \leq \int_0^1 0\, dx = 0,$$

so $u^1 = u^2$ as desired. □

Let $C_0^2([0,1])$ be the space of two times continuously differentiable functions $u : \mathbb{R} \to \mathbb{R}$ that satisfy $u(0) = u(1) = 0$. (Note: u is not the solution of the PDE above, but a function of one variable.) We use the usual inner product $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$. Let L be the differential operator, defined on $C_0^2([0,1])$, given by

$$(Lu)(x) = -u''(x) + q(x)u(x).$$

Consider the eigenvalue problem

$$Lu = \lambda u, \quad u(0) = u(1) = 0. \tag{9}$$

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4c

Show that L is symmetric, i.e., that

$$\langle Lu, v\rangle = \langle u, Lv\rangle \quad \text{for all } u, v \in C_0^2([0,1]).$$

Solution: Integration by parts and $v(0) = v(1) = u(0) = u(1) = 0$ gives

$$\begin{aligned}
\langle Lu, v\rangle &= \int_0^1 (-u''(x) + q(x)u(x))v(x)dx \\
&= \int_0^1 (-u''v) + \int_0^1 quv \\
&= [-u'v]_0^1 - \int_0^1 (-u'v') + \int_0^1 quv \\
&= \int_0^1 u'v' + \int_0^1 quv \\
&= [uv']_0^1 - \int_0^1 uv'' + \int_0^1 quv \\
&= \int_0^1 u(-v'' + qv) \\
&= \langle u, Lv\rangle.
\end{aligned}$$

□

Problem 5

Let $f(x) = |x|$ for $x \in [-1, 1]$. Let $S_N(f)$ be the usual N'th partial sum of the full Fourier series of f.

5a

Determine if $S_N(f)$ converges to f for each of the three convergence types: (i) pointwise convergence, (ii) mean square convergence, and (iii) uniform convergence. (Here you only need to state some general results from the book that give the right conclusions.)

Solution: This is an example in the book (Example 9.7): The 2-periodic extension of f is continuous, as $f(-1) = f(1)$, and its derivative is piecewise continuous, and a general result (Theorem 9.3) then shows that $S_N(f)$ converges uniformly to f on $[-1, 1]$. Moreover, in general, uniform convergence implies pointwise convergence and mean square convergence (see Propostion 9.1). □