Solutions

1-a

$$u(x,t) = e^{-(x-4t)^2}.$$

This is a special case of Example 1.1 in the book.

1-b

$$u(x,t) = \phi(xe^{-t}) + t.$$

See section 1.4.2 in the book.

2-a

The Fourier coefficients c_k are given by

$$\begin{aligned} c_k &= 2\int_0^1 f(x)\sin(k\pi x)\mathrm{d}x = 2\Big(\int_0^{1/2}\sin(k\pi x)\mathrm{d}x - \int_{1/2}^1 \sin(k\pi x)\mathrm{d}x\Big) \\ &= \frac{2}{k\pi}\Big(1 - 2\cos(k\pi/2) + \cos(k\pi)\Big) = \begin{cases} 0, & k = 4m, \\ 0, & k = 4m-1, \\ \frac{8}{k\pi}, & k = 4m-2, \\ 0, & k = 4m-3. \end{cases} \end{aligned} \tag{1}$$

Hence, the Fourier sine series of f is equal to

$$\begin{aligned} f(x) &= \sum_{k=1}^{\infty} c_k \sin(k\pi x) = \sum_{m=1}^{\infty} \frac{8}{(4m-2)\pi}\sin((4m-2)\pi x) \\ &= \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi}\sin((4m-2)\pi x). \end{aligned} \tag{2}$$

2-b

The solution is

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-(k\pi)^2 t}\sin(k\pi x),$$

with c_k given in (1). This follows from Chapter 3 in the book.

2-c

The solution is

$$u(x,t) = \sum_{k=1}^{\infty} \frac{c_k}{k\pi}\sin(k\pi t)\sin(k\pi x),$$

with c_k given in (1). This follows from Chapter 5 in the book.

2-d

The solution is

$$u(x,y) = \sum_{k=1}^{\infty} \frac{c_k}{\sinh(k\pi)} \sin(k\pi x)\sinh(k\pi y),$$

with c_k given in (1). This follows from Chapter 7 in the book.

2-e

Observe that

$$g'(x) = \begin{cases} 1, & -1 < x < -1/2, \\ -1, & 1/2 < x < 0, \\ 1, & 0 < x < 1/2, \\ -1 & 1/2 < x < 1, \end{cases}$$

which is the odd extension of f to the interval $[-1,1]$. Thus we know from Chapter 8 that their Fourier sine series are equal, i.e. that g' is given by (2) on $[-1,1]$. Since g' is piecewise continuous, and $g(-1) = g(1)$, it follows from Theorem 8.1 that

$$\begin{aligned} g(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) = \frac{a_0}{2} - \sum_{k=1}^{\infty} \frac{c_k}{k\pi} \cos(k\pi x) = \frac{a_0}{2} - \sum_{m=1}^{\infty} \frac{8}{((4m-2)\pi)^2} \cos((4m-2)\pi x) \\ &= \frac{a_0}{2} - \sum_{m=1}^{\infty} \frac{2}{((2m-1)\pi)^2} \cos((4m-2)\pi x). \end{aligned}$$

Note that we still have to find a_0 the usual way,

$$a_0 = 2\int_0^1 g(x)\mathrm{d}x = 4\int_0^{1/2} x\mathrm{d}x = \frac{1}{2}.$$

Since the periodic extension of g is continuous, and g' is piecewise continuous, it follows from Theorem 9.1 that the above Fourier series converges uniformly to g.

3-a

Let $E(t)$ be given by

$$E(t) = \int_0^1 u(x,t)^2 \mathrm{d}x,$$

then, using integration by parts, the Poincaré inequality and $0 \le b(x) \le 1$ we get

$$\begin{aligned} E'(t) &= \int_0^1 2uu_t \mathrm{d}x = 2\int_0^1 u(\epsilon u_{xx} + b(x)u)\mathrm{d}x = 2[\epsilon u u_x]_0^1 - 2\epsilon \int_0^1 u_x^2 \mathrm{d}x + 2\int_0^1 b(x)u^2 \mathrm{d}x \\ &\le -2\epsilon\pi^2 \int_0^1 u^2 \mathrm{d}x + 2\int_0^1 u^2 \mathrm{d}x = 2(1-\epsilon\pi^2)E(t). \end{aligned}$$

It then follows from Grönwall's inequality that

$$E(t) \leq e^{2(1-\epsilon\pi^2)t} E(0) \to 0, \quad \text{as } t \to \infty,$$

so long as $1 - \epsilon\pi^2 < 0$, which is equivalent to $\epsilon > \pi^{-2}$.

3-b

Let $f(x) = \sin(\pi x)$. Then $u(x,t) = \sin(\pi x)$ for all t.

3-c

This is true because the first Fourier coefficient of f is equal to zero. To see this let f be any function of the form

$$f(x) = \sum_{k=2}^{\infty} c_k \sin(k\pi x),$$

then from a standard separation of variables argument we have

$$u(x,t) = \sum_{k=2}^{\infty} c_k e^{(1-\epsilon(k\pi)^2)t} \sin(k\pi x).$$

Using Parseval's identity it then follows that

$$\int_0^1 u(x,t)^2 \mathrm{d}x = \frac{1}{2} \sum_{k=2}^{\infty} c_k^2 \, e^{2(1-\epsilon(k\pi)^2)t} \to 0, \quad \text{as } t \to \infty$$

whenever $1 - \epsilon(k\pi)^2 < 0$ for all $k = 2, 3, \ldots$. This holds for $\epsilon > 1/(4\pi^2)$.