UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF 3360 — Introduction to partial differential equations

Day of examination: June 13, 2016

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let $a \geq 0$ and consider the problem

$$\begin{aligned} u_t &= u_{xx}, && x \in (0,1),\ t > 0, \\ u(0,t) &= u(1,t) = a, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned} \tag{1}$$

1a

Solve equation (1) when $a = 0$ and $f(x) = \sin(2\pi x) - \sin(5\pi x)$. (You may use a general formula for the solution, or do the computations.)

1b

Find the formal solution of equation (1) when $f(x) = 1$ and $a \geq 0$.

Problem 2

Consider the initial-boundary value problem of the form

$$\begin{cases} u_t + au_x = 0, & x \in [0,1],\ t > 0, \\ u(0,t) = t, & t > 0, \\ u(x,0) = f(x), & x \in [0,1] \end{cases} \tag{2}$$

where $a > 0$ is a constant, and f is a given continuously differentiable function.

2a

Use the method of characteristics to show that any solution of (2) satisfies

$$u(x,t) = \begin{cases} f(x-at) & \text{for} \quad x > at, \\ t - x/a & \text{for} \quad x < at \end{cases} \tag{3}$$

(Continued on page 2.)

and where we assume $x \in [0, 1]$ and $t \geq 0$.

Let $u(x, t)$ be as in (3), and define

$$u(x,t) = 0 \quad \text{when } x = at,\ x \in [0,1]. \tag{4}$$

2b

Give conditions on f at $x = 0$ such that u, given by (3) and (4), is continuous. Moreover, prove that u satisfies

$$|u(x,t)| \leq \max\{t, M\} \quad x \in [0,1],\ t > 0. \tag{5}$$

where $M = \max\{|f(x_1)| : x_1 \in [0, 1]\}$.

Problem 3

Let $n \geq 1$, $h = 1/(n + 1)$ and define grid points $(x_j, y_k) = (jh, kh)$ for $0 \leq j, k \leq n+1$. Let v be a grid function, with $v_{j,k} = v(x_j, y_k)$. Recall that v is called a *discrete harmonic function* if

$$L_h v(x_j, y_k) = 0 \qquad 1 \leq j, k \leq n$$

where the finite difference operator L_h is defined by

$$(L_h v)(x_j, y_k) = \frac{1}{h^2}[4v_{j,k} - v_{j+1,k} - v_{j-1,k} - v_{j,k+1} - v_{j,k-1}].$$

3a

Assume that v is a discrete harmonic function. Show that, for each $1 \leq j, k \leq n$,

$$m_{jk} \leq v_{j,k} \leq M_{jk} \tag{6}$$

where

$$\begin{aligned} m_{jk} &= \min\{v_{j+1,k}, v_{j-1,k}, v_{j,k+1}, v_{j,k-1}\}, \\ M_{jk} &= \max\{v_{j+1,k}, v_{j-1,k}, v_{j,k+1}, v_{j,k-1}\}. \end{aligned}$$

3b

Use (6) to establish a maximum principle for discrete harmonic functions, i.e., that the maximum, and the minimum, of such a function v is attained at a grid point on the boundary.

Problem 4

Consider the initial and boundary value problem

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) - q(x)u(x,t), && x \in (0,1),\ t > 0, \\ u(0,t) &= u(1,t) = 0, && t > 0, \\ u(x,0) &= f(x), && x \in (0,1). \end{aligned} \tag{7}$$

Here q is a given continuous function which satisfies $q(x) \geq 0$ for all $x \in [0, 1]$. Let $u = u(x, t)$ be a solution of (7), and define, for each $t \geq 0$, the energy

$$E(t) = \int_0^1 u^2(x,t)dx.$$

(Continued on page 3.)

4a

Use energy arguments to show that

$$E(t) \leq \int_0^1 f^2(x)dx \quad \text{for } t > 0. \tag{8}$$

Hint: You may assume that it is possible to interchange the order of differentiation and integration.

4b

Use the inequality in (8) to show that the PDE in (7) has at most one solution.

Let $C_0^2([0,1])$ be the space of two times continuously differentiable functions $u : \mathbb{R} \to \mathbb{R}$ that satisfy $u(0) = u(1) = 0$. (Note: u is not the solution of the PDE above, but a function of one variable.) We use the usual inner product $\langle u, v\rangle = \int_0^1 u(x)v(x)dx$. Let L be the differential operator, defined on $C_0^2([0,1])$, given by

$$(Lu)(x) = -u''(x) + q(x)u(x).$$

Consider the eigenvalue problem

$$Lu = \lambda u, \quad u(0) = u(1) = 0. \tag{9}$$

4c

Show that L is symmetric, i.e., that

$$\langle Lu, v\rangle = \langle u, Lv\rangle \quad \text{for all } u, v \in C_0^2([0,1]).$$

Problem 5

Let $f(x) = |x|$ for $x \in [-1, 1]$. Let $S_N(f)$ be the usual N'th partial sum of the full Fourier series of f.

5a

Determine if $S_N(f)$ converges to f for each of the three convergence types: (i) pointwise convergence, (ii) mean square convergence, and (iii) uniform convergence. (Here you only need to state some general results from the book that give the right conclusions.)

Good luck!