MAT3360: Mandatory assignment #2, spring 2018

To be handed in by April 5., 14:30

You must hand in *one* (preferably .pdf) file containing your answers as well as commented scripts which actually compile and work.

You must also use “Devilry”.

Exercise 1. Consider the operator $L[u] = -\varepsilon u'' + u'$ acting on functions in $C_0^2((0,1))$, where $\varepsilon > 0$ is a constant. Recall that $u \in C_0^2((0,1))$ if u is continuous in $[0,1]$ with $u(0) = u(1) = 0$, and $u(x)$ is twice continuously differentiable for $x \in (0,1)$.

a) Find the eigenvalues and the eigenfunctions of L, i.e., all solutions of

$$L[u](x) = -\varepsilon u''(x) + u'(x) = \lambda u(x), \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

Possible solution: We have that $\varepsilon u'' - u' + \lambda u = 0$. Guessing a solution on the form $u(x) = e^{\sigma x}$ we obtain the equation

$$\varepsilon \sigma^2 - \sigma + \lambda = 0,$$

and

$$\sigma = \frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1 - 4\lambda\varepsilon}.$$

If $4\lambda\varepsilon \leq 1$, we get solutions on the form

$$u(x) = e^{ax}(c_1 e^{bx} + c_2 e^{-bx}),$$

with a and b real numbers. The boundary conditions the imply that $c_1 = c_2 = 0$. Hence $4\lambda\varepsilon > 1$. Then we get solutions on the form

$$u(x) = e^{ax}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right), \qquad \begin{cases} a = \frac{1}{2\varepsilon}, \\ \beta = \frac{\sqrt{4\lambda\varepsilon - 1}}{2\varepsilon}. \end{cases}$$

Now the boundary conditions imply that $\beta = k\pi$ for $k = 1, 2, \ldots$, i.e.,

$$\lambda = \lambda_k = \frac{1}{4\varepsilon}\left(4\varepsilon^2 k^2 \pi^2 + 1\right) = \frac{1}{4\varepsilon} + \varepsilon k^2 \pi^2.$$

Hence we get the eigenvectors and eigenvalues

$$u_k(x) = e^{x/2\varepsilon}\sin(k\pi x), \quad \lambda_k = \frac{1}{4\varepsilon} + \varepsilon k^2\pi^2, \quad k = 1, 2, 3, \ldots$$

b) Define the dual operator to L, L^* by requiring that

$$\langle Lu, v\rangle = \langle u, L^* v\rangle, \quad \text{for all } u, v \text{ in } C_0^2((0,1)).$$

Here $\langle u, v\rangle$ is the inner product on C_0^2 defined by

$$\langle u, v\rangle = \int_0^1 u(x)v(x)\,dx.$$

Find an expression for L^* as a differential operator, and find all its eigenvectors u_k^* and eigenvalues λ_k^*.

Possible solution: We find that

$$\begin{aligned}\langle Lu, v\rangle &= \int_0^1 -\varepsilon u''(x)v(x) + u'(x)v(x)\,dx \\ &= \int_0^1 -u(x)\varepsilon v''(x) - u(x)v'(x)\,dx,\end{aligned}$$

using integration by parts. Since this should hold for all $u \in C_0^2$, we have $L^* v = -\varepsilon v'' - v'$.

Regarding the eigenvalues and the eigenfunctions, only the sign of a will change, thus we have that

$$u_k^*(x) = e^{-x/2\varepsilon} \sin(k\pi x), \quad \lambda_k^* = \lambda_k.$$

c) Show that

$$\langle u_k, u_\ell^* \rangle = \begin{cases} \frac{1}{2} & \ell = k, \\ 0 & \ell \neq k. \end{cases}$$

Possible solution: Since $\lambda_k = \lambda_k^*$, we get

$$\lambda_k \langle u_k, u_\ell^* \rangle = \langle L u_k, u_\ell^* \rangle = \langle u_k, L^* u_\ell^* \rangle = \lambda_\ell \langle u_k, u_\ell^* \rangle.$$

From this we get orthogonality, that $\langle u_k, u_k^* \rangle = \frac{1}{2}$ is straightforward to show.

Exercise 2. Let $u = u(x,t)$ satisfy

$$\begin{aligned} u_t + u_x &= \varepsilon u_{xx}, \quad t > 0, \quad x \in (0,1) \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= f(x). \end{aligned} \tag{1}$$

a) Find a formal solution to this problem. (**Hint**: Use separation of variables.)

Possible solution: Writing $u = XT$ leads to the two equations

$$T' = -\lambda T, \quad \varepsilon X'' - X = -\lambda X,$$

with $X(0) = X(1) = 0$. Then $X = X_k(x) = e^{x/2\varepsilon} \sin(k\pi x)$ and $\lambda = \lambda_k = 1/4\varepsilon + \varepsilon k^2 \pi^2$. Then

$$T = T_k(t) = \exp\left(-\left(\frac{1}{4\varepsilon} + \varepsilon k^2 \pi^2 \right) t \right)$$

Then a formal solution is given by

$$u(x,t) = \sum_{k=1}^{\infty} c_k \exp\left(-\left(\frac{1}{4\varepsilon} + \varepsilon k^2 \pi^2 \right) t \right) e^{x/2\varepsilon} \sin(k\pi x).$$

For the initial condition to be satisfied we must have

$$c_k = 2\langle f, X_k^* \rangle = 2 \int_0^1 e^{-x/2\varepsilon} f(x) \sin(k\pi x)\, dx.$$

b) Let $E(t) = \frac{1}{2} \int_0^1 u^2\, dx$. Show that $E(t) \le E(0)$, and conclude that (1) has at most one solution which is a *classical solution*, i.e., twice continuously differentiable in x and once continuously differentiable in t for $(x,t) \in [0,1] \times [0,\infty)$.

Possible solution: Multiplying the equation by u we get

$$\frac{1}{2} \frac{\partial}{\partial t} u^2 + \frac{1}{2} \frac{\partial}{\partial x} u^2 = \varepsilon u u_{xx}.$$

Integrating between 0 and 1,

$$\frac{1}{2} \frac{d}{dt} \int_0^1 u^2(x,t)\, dx + \frac{1}{2} u^2(x,t) \Big|_{x=0}^{x=1} = -\varepsilon \int_0^1 (u_x(x,t))^2\, dx \le 0.$$

Thus $E'(t) \le 0$ and the answer follows. If u and v are two classical solutions, then $\int_0^1 (u-v)^2\,dx \le 0$ from which follows that $u = v$.

c) Show that for $t \ge 0$

$$\min_{x\in[0,1]} f(x) \le u(x,t) \le \max_{x\in[0,1]} f(x),$$

when u is a classical solution of (1).

Possible solution: For $\sigma > 0$, set $u^\sigma = u + \sigma e^{-t}$. Then $u_t^\sigma = u_t - \sigma e^{-t}$, $u_x^\sigma = u_x$ and $u_{xx}^\sigma = u_{xx}$. Therefore $u_t^\sigma + u_x^\sigma = \varepsilon u_{xx}^\sigma - \sigma e^{-t}$. Assume that $(\hat{x}, \hat{t})$, $\hat{t} > 0$, are such that

$$u^\sigma(\hat{x},\hat{t}) \ge u^\sigma(x,t) \quad \text{for all } x \in [0,1] \text{ and all } t \in [0,\hat{t}].$$

Then

$$u_t^\sigma(\hat{x},\hat{t}) \ge 0, \quad u_x^\sigma(\hat{x},\hat{t}) = 0 \quad \text{and} \quad u_{xx}^\sigma(\hat{x},\hat{t}) \le 0.$$

It follows that

$$0 \le u_t^\sigma(\hat{x},\hat{t}) = u_t^\sigma(\hat{x},\hat{t}) + u_x^\sigma(\hat{x},\hat{t}) = \varepsilon u_{xx}^\sigma - \sigma e^{-\hat{t}} \le -\sigma e^{-\hat{t}} < 0.$$

This is a contradiction, and $u^\sigma(x,t) < \max_x u^\sigma(x,0) = \max_x f(x) + \sigma$. This is the same as

$$\max_x u(x,t) + \sigma e^{-t} < \max_x f(x) + \sigma.$$

This holds for all $\sigma > 0$, hence we can let $\sigma \searrow 0$, to obtain $\max_x u(x,t) \le \max_x f(x)$. By using $\sigma < 0$ we obtain the lower bound $\min_x u(x,t) \ge \min_x f(x)$.

d) Find the solution to (1) if

$$f(x) = e^{x/2\varepsilon}\sin(\pi x).$$

Possible solution: We see that only c_1 is different from zero. Furthermore $c_1 = 1$. In this case

$$u(x,t) = e^{-a_\varepsilon t} e^{x/2\varepsilon}\sin(\pi x), \qquad a_\varepsilon = \frac{1}{4\varepsilon} + \varepsilon\pi^2.$$

Exercise 3. Consider the difference scheme for (1):

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} + \frac{v_j^m - v_{j-1}^m}{\Delta x} = \varepsilon\frac{v_{j+1}^{m+1} - 2v_j^{m+1} + v_{j-1}^{m+1}}{\Delta x^2}, \quad \text{for } j = 1,\dots,n,\ m \ge 0, \tag{2}$$
$$v_j^0 = f(j\Delta x).$$

for $j = 1,\dots,n$, $\Delta t > 0$ and $\Delta x = 1/(n+1)$ with $v_0^m = v_{n+1}^m = 0$.

a) Find a relation between Δt and Δx such that this relation implies the discrete maximum principle

$$\min_j f(j\Delta x) \le v_j^m \le \max_j f(j\Delta x), \quad \text{for all } j \in 1,\dots,n \text{ and } m \ge 0.$$

Possible solution: Assume that J is such that $v_J^{m+1} \ge v_j^{m+1}$ for all $j = 1,\dots,n$. Then $2v_J^{m+1} \ge v_{J-1}^{m+1} + v_{J+1}^{m+1}$, and hence

$$v_J^{m+1} \le v_J^m - \frac{\Delta t}{\Delta x}\left(v_J^m - v_{J-1}^m\right) = \left(1 - \frac{\Delta t}{\Delta x}\right)v_J^m + \frac{\Delta t}{\Delta x}v_{J-1}^m.$$

If $\Delta t/\Delta x \le 1$, then this is a *convex combination* of v_J^m and v_{J-1}^m. Therefore

$$\max_j v_j^{m+1} = v_J^{m+1} \le \max\left\{v_J^m, v_{J-1}^m\right\} \le \max_j v_j^m.$$

The analogous inequality for $\min_j v_j^{m+1}$ is similarly established. So in order to have stability we must choose $\Delta t \le \Delta x$.

b) Set $v^m = (v_1^m, \dots, v_n^m)^T$, find matrices A and B such that the scheme can be written

$$(I + \varepsilon \Delta t A) v^{m+1} = (I + \Delta t B) v^m.$$

Possible solution: We see that

$$A = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \qquad B = \frac{1}{\Delta x} \begin{pmatrix} -1 & 0 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 1 & -1 \end{pmatrix}$$

c) Implement a routine that given v^0, ϵ and T computes v^m with $m\Delta t = T$. You do not have to write a solver for the linear equation, and can use any ready made solver, such as Matlab's "\".

Possible solution: Here is my code:

```
function v=mysolver(epsil,f,T)
n=length(f)-2; % f includes endpoints
dx=1/(n+1);  dt=dx;
M=ceil(T/dt);
dt=T/M;

mu=epsil*dt/dx.^2; lambda=dt/dx;
% Assembling the matrices
e=ones(n,1); A=spdiags([-e 2*e -e],-1:1,n,n);
AA=speye(n)+mu*A;
B=spdiags([e -e], -1:0,n,n);
BB=speye(n)+lambda*B;
v=f(2:end-1);
for m=1:M
    v=AA\BB*v;
end
v=[0;v;0]; % Padding the endpoints
```

d) Test your code using the exact solution from **2d)**. Use $\varepsilon = 0.125$, $T = 1$ and $n = 16, 32, 64, \dots, 512$, to estimate the convergence rate measuring the error by

$$\max_j \left| u(j\Delta x, 1) - v_j^m \right|, \quad \text{where } m\Delta t = 1.$$

Possible solution: I implemented the exact solution:

```
function y=exactsol(x,epsil,t)
y=exp(-(1/(4*epsil)+epsil*pi^2)*t)*exp(x/(2*epsil)).*sin(pi*x);
```

Measuring the convergence rate:

```
>> T=1; e=0.125;
>> n=16; for i=1:6, h=1/(n+1); x=(0:n)'*h; f=exactsol(x,e,0); ...
     v=mysolver(e,f,T); err(i)=max(abs(v-exactsol(x,e,T))); n=2*n; end;
>> rate=log(err(2:end)./err(1:end-1))./log(2)
```

```
    rate =

       -0.7863   -0.8903   -0.9444   -0.9720   -0.9860

>>
```

As expected, since the approximation to the time (and space) first derivatives are first order, we get a first order convergence.