MAT3360: Mandatory assignment #2, spring 2019
Suggested solution

Exercise 1.

(a) We differentiate $E_u(t)$ with respect to t:

$$\begin{aligned}
E_u'(t) &= \int_0^1 \frac{\partial}{\partial t}\big(u(x,t)^2\big)\,dx = 2\int_0^1 u(x,t)u_t(x,t)\,dx && \text{(chain rule)}\\
&= -2c\int_0^1 u(x,t)u_x(x,t)\,dx && \text{(by (1))}\\
&= -c\int_0^1 \frac{\partial}{\partial x}\big(u(x,t)^2\big)\,dx && \text{(chain rule)}\\
&= -c\Big[u(x,t)^2\Big]_{x=0}^1 && \text{(integration by parts)}\\
&= 0 && \text{(using boundary conditions).}
\end{aligned}$$

Hence, $E_u(t) = E_u(0)$ for all $t \geqslant 0$.

(b) Let u and v be two solutions of (1) with the same initial data f. Then $w := u - v$ is also a solution of (1) with initial data $w(x,0) = 0$. From **(a)** we see that $E_w(t) = E_w(0) = 0$ for all $t \geqslant 0$, that is,

$$\int_0^1 w(x,t)^2\,dx = 0.$$

But then $w(x,t) \equiv 0$, and hence $u \equiv v$.

(c) If ϕ is any twice differentiable function then both the "forward" and "backward differences"

$$\phi'(y) \approx \frac{\phi(y+h)-\phi(y)}{h} \qquad \text{and} \qquad \phi'(y) \approx \frac{\phi(y)-\phi(y-h)}{h}$$

for $h > 0$ are first-order approximations of the derivative of ϕ. Both (2a) and (2b) use the forward difference approximation

$$u_t(x_j,t_m) \approx \frac{u(x_j,t_m+\Delta t)-u(x_j,t_m)}{\Delta t} = \frac{u(x_j,t_{m+1})-u(x_j,t_m)}{\Delta t},$$

while for the space derivative, (2a) uses a "backward difference" and (2b) a "forward difference". The boundary conditions are set as $v_0^m = v_{n+1}^m$ because $x_0 = 0$ and $x_1 = 1$ (compare with (1)).

(d) We make the ansatz that $v_j^m = a^m e^{ik\pi x_j}$ for some integer k. From the boundary condition we see that we need $1 = v_0^m = v_{n+1}^m = e^{ik\pi}$, and therefore $k = 2\ell$ for some $\ell \in \mathbb{Z}$. We see moreover that we can restrict ourselves to values $k = 2\ell \in \{-n,\dots,n+1\}$, because any values of k outside of this set yield the same function $e^{ik\pi x_j}$.

Inserting our ansatz into (2a) gives

$$\frac{a^{m+1}-a^m}{\Delta t}e^{ik\pi x_j} + ca^m\frac{e^{ik\pi x_j}-e^{ik\pi x_{j-1}}}{\Delta x} = 0.$$

Dividing by $a^m e^{ik\pi x_j}$ on both sides and solving for a gives

$$a = 1 - cr + cre^{-ik\pi\Delta x}, \qquad r = \frac{\Delta t}{\Delta x}.$$

If we assume that $cr \leqslant 1$ (recall that $c > 0$) then

$$|a| \leqslant |1-cr| + |cre^{-ik\pi\Delta x}| = 1 - cr + cr = 1,$$

which is what we want. Hence, the CFL condition $c\frac{\Delta t}{\Delta x} \leqslant 1$ implies von Neumann stability.

If we do the same for (2b) then we get

$$a = 1 + cr - cre^{ik\pi\Delta x}, \qquad r = \frac{\Delta t}{\Delta x}.$$

Whenever k is such that $e^{ik\pi\Delta x} \neq 1$ we have $\Re(e^{ik\pi\Delta x}) < 1$ (where $\Re(z)$ denotes the real part of $z \in \mathbb{C}$), so

$$\Re(a) = 1 + cr - cr\Re(e^{ik\pi\Delta x}) > 1 + cr - cr = 1,$$

so in particular, $|a| > 1$. This holds regardless of $\Delta x, \Delta t > 0$, so the method (2b) is unconditionally unstable.

(e) Assume for simplicity that n is an even number. Expand the numerically computed solution v_j^m as a discrete Fourier series,

$$v_j^m = \sum_{\ell=-n/2}^{n/2} c_{j,\ell,m} X_\ell(x_j), \qquad X_\ell(x_j) = e^{2i\ell\pi x_j}$$

for some $c_{j,\ell,m} \in \mathbb{C}$. From **(d)** we know that we can write $c_{j,\ell,m} = c_\ell a_\ell^m$ for some $a_\ell, c_\ell \in \mathbb{C}$, and that the CFL condition implies that $|a_\ell| \leqslant 1$. Writing

$$\langle u,\, v\rangle_{\Delta x} = \Delta x \sum_{j=0}^{n} u_j \overline{v_j}$$

(where $\overline{v_j}$ is the complex conjugate of v_j), we have the orthogonality property

$$\langle X_\ell,\, X_{\ell'}\rangle_{\Delta x} = \begin{cases} 1 & \ell = \ell' \\ 0 & \text{else.} \end{cases}$$

Moreover, $\tilde{E}_v(t_m) = \langle v^m,\, v^m\rangle_{\Delta x}$. Hence,

$$\begin{aligned} \tilde{E}_v(t_m) &= \langle v^m,\, v^m\rangle_{\Delta x} = \langle \textstyle\sum_\ell c_\ell a_\ell^m X_\ell,\, \sum_{\ell'} c_{\ell'} a_{\ell'}^m X_{\ell'}\rangle_{\Delta x} \\ &= \sum_{\ell,\ell'} c_\ell a_\ell^m \overline{c_{\ell'}}\, \overline{a_{\ell'}^m} \langle X_\ell,\, X_{\ell'}\rangle_{\Delta x} \\ &= \sum_\ell |c_\ell|^2 |a_\ell|^{2m} \end{aligned}$$

by the orthogonality of the X_ℓ functions. Since $|a_\ell^m| \leqslant 1$, we see that the energy $\tilde{E}_v(t_m)$ decreases when m increases.

(f) (1) is a linear, first-order transport equation, so by the theory of characteristics we can write

$$u(x,t) = f(x - ct)$$

(we ignore the boundary conditions here). Since $c > 0$, we see that the solution $u(x,t)$ depends on values of f *to the left* of the point x. Since the upwind method (2a) takes values of v to the left of the point x_j, this is a reasonable discretization of (1). The method (2b), on the other hand, takes values of v *to the right* of x_j, which does not reflect the behavior of the exact solution.

Exercise 2.

(a) Let $y(t)$ be the characteristic

$$y'(t) = c(y(t)), \qquad y(0) = y_0 \in \mathbb{R}.$$

Then $\frac{d}{dt}u(y(t),t) = 0$, so $u(y(t),t) = f(y_0)$ for any $t \geqslant 0$. Since c is smooth and bounded, there is for every point (x,t) some value $y_0 \in \mathbb{R}$ such that the corresponding characteristic goes through this point, that is, $y(t) = x$. Thus,

$$u(x,t) = f(y_0).$$

In particular, we find that u must be upper-bounded by the maximum value of f, and lower-bounded by its minimum value, which is precisely (5).

(b) If u and v are solutions of (4) then $w := u - v$ is also a solution, with $w(x,0) = f(x) - g(x)$. Hence, by **(a)** we find that

$$\inf_{y\in\mathbb{R}} f(y) - g(y) \leqslant w(x,t) \leqslant \sup_{y\in\mathbb{R}} f(y) - g(y).$$

Hence,

$$\begin{aligned} |w(x,t)| &\leqslant \max\Big(\sup_{y\in\mathbb{R}} f(y) - g(y),\ -\inf_{y\in\mathbb{R}} f(y) - g(y)\Big) \\ &= \max\Big(\sup_{y\in\mathbb{R}} f(y) - g(y),\ \sup_{y\in\mathbb{R}} g(y) - f(y)\Big) \\ &= \sup_{y\in\mathbb{R}} |f(y) - g(y)| = \|f - g\|_\infty. \end{aligned}$$

(c) We write the scheme (6) as

$$v_j^{m+1} = (1 - c_j r)v_j^m + c_j r v_{j-1}^m, \qquad r = \frac{\Delta t}{\Delta x}.$$

Since $c \geqslant 0$ we automatically get $c_j r \geqslant 0$, and if $\|c\|_\infty r \leqslant 1$ then also $c_j r \leqslant 1$. Hence, v_j^{m+1} is a convex combination of v_j^m and v_{j-1}^m. Hence, if

$$\inf_{k\in\mathbb{Z}} f(x_k) \leqslant v_j^m \leqslant \sup_{k\in\mathbb{Z}} f(x_k)$$

then v_j^{m+1} will also satisfy this bound. This is clearly true when $m = 0$, so by induction, it holds for all $m \geqslant 0$.

(d) The numerical method (6) is linear, so if v and w are numerically computed solutions then $e_j^m := v_j^m - w_j^m$ is also a numerically computed solution, with initial data $e_j^0 = f(x_j) - g(x_j)$. From the maximum principle in **(c)** we then find that

$$\inf_{k\in\mathbb{Z}} f(x_k) - g(x_k) \leqslant v_j^m - w_j^m \leqslant \sup_{k\in\mathbb{Z}} f(x_k) - g(x_k).$$

Arguing as in **(b)**, we conclude that

$$|v_j^m - w_j^m| \leqslant \sup_{k\in\mathbb{Z}} |f(x_k) - g(x_k)| \leqslant \|f - g\|_\infty,$$

and taking the supremum of this inequality over all $j \in \mathbb{Z}$ gives the desired stability bound.