MAT3360 — Introduction to Partial Differential Equations

Mandatory assignment 2 of 2

Submission deadline

Monday 8 April 2019 at 14:30 through Canvas.

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Consider the transport equation with periodic boundary conditions

$$\begin{cases} u_t + cu_x = 0 & x \in (0,1), t > 0 \\ u(0,t) = u(1,t) & t > 0 \\ u(x,0) = f(x) & x \in (0,1) \end{cases} \tag{1}$$

where $c > 0$ is a given number and $f : [0,1] \to \mathbb{R}$ is a given continuous function.

(a) Show that the *energy* $E_u(t) = \int_0^1 u(x,t)^2\,dx$ of any solution u of (1) is constant over time.

(b) Use **(a)** to prove that there exists at most one solution of (1) for a given initial data f.

(c) We now consider the following two finite difference methods for (1):

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} + c\frac{v_j^m - v_{j-1}^m}{\Delta x} = 0 \qquad \text{for } j = 1, \dots, n+1,\ m = 0, 1, \dots \tag{2a}$$

and

$$\frac{v_j^{m+1} - v_j^m}{\Delta t} + c\frac{v_{j+1}^m - v_j^m}{\Delta x} = 0 \qquad \text{for } j = 0, \dots, n,\ m = 0, 1, \dots, \tag{2b}$$

both using the boundary- and initial conditions

$$\begin{cases} v_0^m = v_{n+1}^m & m = 0, 1, \dots \\ v_j^0 = f(x_j) & j = 1, \dots, n+1. \end{cases} \tag{3}$$

Explain the derivation of both of these method.

(d) Derive a *CFL condition* which ensures that the method (2a),(3) is stable in the sense of von Neumann. Show that the method (2b),(3) is *unconditionally unstable* — that is, it is unstable under any CFL condition.

Hint: Recall that $c > 0$.

Hint: Write the numerical solution as $v_j^m = a^m e^{ik\pi x_j}$ for some $a \in \mathbb{C}$ (the *amplification factor*) and $k \in \{-n+1, \dots, n-1\}$ (the *wave number*) and find a condition on Δt which ensures that $|a| \leqslant 1$.

(e) Show that the von Neumann stability of the method (2a),(3) implies that the discrete energy $\widetilde{E}_v(t_m) = \Delta x \sum_{j=0}^n |v_j^m|^2$ does not increase over time.

(f) Derive an explicit solution formula for (1). Use this expression to explain why the *upwind method* (2a) is a reasonable discretization, while (2b) is a bad discretization of the equation. To simplify matters you can ignore the periodic boundary conditions.

Problem 2. We now consider the Cauchy problem for the transport equation with a *nonconstant* velocity,

$$\begin{cases} u_t(x,t) + c(x)u_x(x,t) = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{4}$$

where $c, f \in C^1(\mathbb{R})$ are given differentiable and bounded functions. For this equation the energy is no longer preserved over time, and the von Neumann stability analysis is not applicable.

(a) Show that the solution of (4) satisfies the maximum principle

$$\inf_{y\in\mathbb{R}} f(y) \leqslant u(x,t) \leqslant \sup_{y\in\mathbb{R}} f(y) \tag{5}$$

for all $x \in \mathbb{R}$ and $t \geqslant 0$.

Hint: Use the solution formula for (4).

(b) Show that (4) is stable, in the following sense: Any two solutions u and v of (4) with initial data f and g, respectively, satisfy

$$\sup_{x\in\mathbb{R}} \big|u(x,t) - v(x,t)\big| \leqslant \|f - g\|_\infty \qquad \forall\, t \geqslant 0.$$

Explain why there can be at most one solution of (4).

(c) Assume now that $c(x) \geqslant 0$ for all $x \in \mathbb{R}$, and consider the following finite difference scheme for (4):

$$\begin{cases} \frac{v_j^{m+1} - v_j^m}{\Delta t} + c_j \frac{v_j^m - v_{j-1}^m}{\Delta x} = 0 & \text{for } j \in \mathbb{Z},\ m = 0, 1, \dots \quad \text{for } j \in \mathbb{Z} \\ v_j^0 = f(x_j) \end{cases} \tag{6}$$

where $c_j = c(x_j)$. Show that the CFL condition

$$\frac{\Delta t}{\Delta x}\|c\|_\infty \leqslant 1$$

guarantees that the numerical solution v_j^m satisfies a maximum principle similar to (5).

(d) Show that if v_j^m and w_j^m are two numerically computed solutions with initial data f and g, respectively, then

$$\sup_{j \in \mathbb{Z}} \left| v_j^m - w_j^m \right| \leqslant \|f - g\|_\infty \qquad \forall\, m \geqslant 0.$$