

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3360 — Introduction to partial differential equations

Day of examination: Friday, June 11, 2021

Examination hours: 09:00–13:00

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: Any

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 15%)

Consider the PDE

$$\begin{cases} u_t + (1 + x^2)u_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(x, 0) = \frac{1}{1+x^2}. \end{cases}$$

Find a solution to this initial value problem.

Løsningsforslag: The characteristic equation is

$$x' = (1 + x^2), \quad x(0) = x_0,$$

with solution

$$x_0 = \tan(\arctan(x) - t).$$

Hence a solution of the PDE is

$$u(x, t) = \frac{1}{1 + (\tan(\arctan(x) - t))^2}.$$

Problem 2 (weight 25%)

Consider the function f defined by

$$f(x) = \begin{cases} \frac{\sin(\pi x)}{x} & x \neq 0, \\ \pi & x = 0. \end{cases}$$

We have that the full Fourier series of f is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) + b_k \sin(k\pi x).$$

(Continued on page 2.)

2a

Explain why the Fourier series converges uniformly to f for $x \in [-1, 1]$, and converges uniformly to a function g for $x \in \mathbb{R}$. Draw the graph of g for $x \in [-3, 3]$.

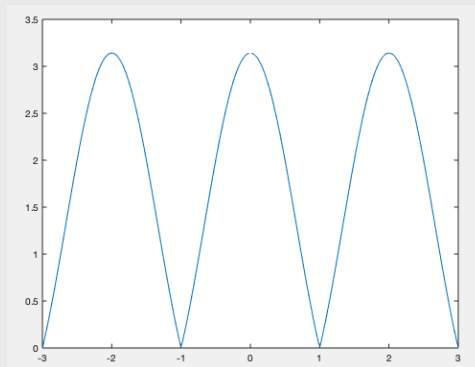
Løsningsforslag: We have that f is continuous on $[-1, 1]$, since

$$\lim_{x \rightarrow 0} f(x) = \pi.$$

f' is continuous on $[-1, 1]$ since

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - \pi}{h} = \lim_{h \rightarrow 0} \frac{\sin(\pi h) - \pi h}{h^2} = -\frac{\pi^2}{2} = \lim_{x \rightarrow 0} f'(x).$$

Hence the Fourier series for f' converges pointwise (to its periodic extension). Then the Fourier series for f will converge uniformly to the periodic extension of f .

**2b**

Show that $b_k = 0$ and that

$$a_k = \int_{k-1}^{k+1} \frac{\sin(\pi x)}{x} dx, \quad k = 0, 1, 2, 3, \dots$$

Løsningsforslag: $b_k = 0$ since f is even. Then

$$\begin{aligned} a_k &= 2 \int_0^1 \frac{\sin(\pi x) \cos(k\pi x)}{x} dx \\ &= \int_0^1 \frac{\sin((k+1)\pi x) - \sin((k-1)\pi x)}{x} dx \\ &= \int_0^{k+1} \frac{\sin(\pi y)}{y} dy - \int_0^{k-1} \frac{\sin(\pi y)}{y} dy \\ &= \int_{k-1}^{k+1} \frac{\sin(\pi y)}{y} dy. \end{aligned}$$

(Continued on page 3.)

2c

Use the Fourier series of f to calculate the improper integral

$$\int_0^\infty \frac{\sin(\pi x)}{x} dx.$$

Løsningsforslag: We know that

$$\begin{aligned} \pi &= f(0) = \lim_{N \rightarrow \infty} \frac{a_0}{2} + \sum_{k=1}^N a_k \\ &= \lim_{N \rightarrow \infty} \left(\int_0^1 \frac{\sin(\pi x)}{x} dx + \sum_{k=1}^N \int_{k-1}^{k+1} \frac{\sin(\pi x)}{x} dx \right) \\ &= \int_0^1 \frac{\sin(\pi x)}{x} dx + \lim_{N \rightarrow \infty} \sum_{k=0}^{N-1} \int_k^{k+1} \frac{\sin(\pi x)}{x} dx + \sum_{k=1}^N \int_k^{k+1} \frac{\sin(\pi x)}{x} dx \\ &= \lim_{N \rightarrow \infty} \int_0^N \frac{\sin(\pi x)}{x} dx + \lim_{N \rightarrow \infty} \int_0^{N+1} \frac{\sin(\pi x)}{x} dx \\ &= 2 \int_0^\infty \frac{\sin(\pi x)}{x} dx. \end{aligned}$$

Therefore $\int_0^\infty \frac{\sin(\pi x)}{x} dx = \pi/2$.

Problem 3 (weight 30%)

Let $Q(x, t)$ be a continuous function $Q : [0, 1] \times [0, T] \rightarrow \mathbb{R}$ such that the map $x \mapsto Q(x, t)$ is in $C_0^2((0, 1))$ for all $t \in [0, T]$. For $k = 1, 2, 3, \dots$ define $X_k(x) = \sin(k\pi x)$.

3a

Define

$$u_N(x, t) = 2 \int_0^t \int_0^1 \sum_{k=1}^N Q(y, s) X_k(x) X_k(y) e^{-(k\pi)^2(t-s)} dy ds.$$

Show that u_N is a solution to the boundary value problem

$$\begin{cases} \frac{\partial}{\partial t} u_N - \frac{\partial^2}{\partial x^2} u_N = Q_N & t \in (0, T], x \in (0, 1), \\ u_N(0, t) = u_N(1, t) = 0 & t > 0, \\ u_N(x, 0) = 0, \end{cases}$$

where

$$Q_N(x, t) = 2 \sum_{k=1}^N X_k(x) \int_0^1 X_k(y) Q(y, t) dy.$$

(Continued on page 4.)

Løsningsforslag: We calculate

$$\frac{\partial}{\partial t} u_N = 2 \int_0^1 \sum_{k=1}^N Q(y, t) X_k(x) X_k(y) dy ds - 2 \int_0^t \int_0^1 \sum_{k=1}^N Q(y, s) X_k(x) X_k(y) (k\pi)^2 e^{-(k\pi)^2(t-s)} dy ds$$

and

$$\frac{\partial^2}{\partial x^2} u_N = -2 \int_0^t \int_0^1 \sum_{k=1}^N Q(y, s) (k\pi)^2 X_k(x) X_k(y) e^{-(k\pi)^2(t-s)} dy ds.$$

Hence u_N satisfies the differential equation. It is easy to see that the initial and boundary conditions are satisfied.

3b

Show that $Q_N \rightarrow Q$ uniformly in $[0, 1] \times [0, T]$.

Løsningsforslag: We recognise Q_N as the partial sum of the Fourier expansion of $Q(\cdot, t)$

$$Q_N(\cdot, t) = \sum_{k=1}^N \frac{\langle Q(\cdot, t), X_k \rangle}{\|X_k\|^2} X_k.$$

Since $Q(\cdot, t) \in C_0^2((0, 1))$ we know that its Fourier series converges uniformly to $Q(x, t)$. This happens for every t in the compact set $[0, T]$. Hence we have uniform convergence for $(x, t) \in [0, 1] \times [0, T]$.

3c

Assume that there exists a smooth solution u to the problem

$$\begin{cases} \frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = Q & t \in (0, T], x \in (0, 1), \\ u(0, t) = u(1, t) = 0 & t > 0, \\ u(x, 0) = 0, \end{cases}$$

Set $E(t) = \|u(\cdot, t)\|$, where $\|\cdot\|$ denotes the mean square norm.

Show that

$$E(t) \leq \int_0^t \|Q(\cdot, s)\| ds.$$

Løsningsforslag: We multiply with u and integrate over x to find

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 + \|u_x(\cdot, t)\|^2 = \langle u(\cdot, t), Q(\cdot, t) \rangle \leq \|u(\cdot, t)\| \|Q(\cdot, t)\|.$$

Hence

$$\frac{d}{dt} \|u(\cdot, t)\| \leq \|Q(\cdot, t)\|.$$

(Continued on page 5.)

3d

Show that u_N converges in the mean square norm to u as $N \rightarrow \infty$.

Løsningsforslag: We get that

$$\frac{\partial}{\partial t} (u - u_N) - \frac{\partial^2}{\partial x^2} (u - u_N) = (Q - Q_N).$$

Multiply this with $(u - u_N)$ and integrate to get

$$\frac{d}{dt} \|u(\cdot, t) - u_N(\cdot, T)\| \leq \int_0^t \|Q(\cdot, s) - Q_N(\cdot, s)\| ds \leq \|Q - Q_N\|_\infty t.$$

Hence

$$\|u(\cdot, t) - u_N(\cdot, t)\| \leq \frac{t^2}{2} \|Q - Q_N\|_\infty \rightarrow 0,$$

as $N \rightarrow \infty$.

Problem 4 (weight 30%)

Consider the transport equation in the periodic setting

$$\begin{cases} u_t + u_x = 0, & t > 0, x \in [0, 1], \\ u(0, t) = u(1, t) \\ u(x, 0) = f(x), \end{cases} \quad (1)$$

where f is a given smooth periodic function with period 1.

Consider also the difference scheme

$$L_{\Delta x} v_j^m := \frac{v_j^{m+1} - \frac{1}{2}(v_{j+1}^m + v_{j-1}^m)}{\Delta t} + \frac{v_{j+1}^m - v_{j-1}^m}{2\Delta x} = 0, \quad m \geq 0, j = 0, 1, \dots, N,$$

and $v_{-1}^m = v_N^m, v_{N+1}^m = v_0^m$. The initial values are given by

$$v_j^0 = f(x_j).$$

Here Δt is a small positive number, $\Delta x = 1/(N+1)$ and $x_j = j\Delta x$. We also define $t^m = m\Delta t$. The scheme is explicit since we can solve for v_j^{m+1} ,

$$v_j^{m+1} = \frac{1}{2} (1 - r) v_{j+1}^m + \frac{1}{2} (1 + r) v_{j-1}^m,$$

with $r = \Delta t / \Delta x$.

4a

Find a condition on r which guarantees that

$$\min_j v_j^m \leq v_j^{m+1} \leq \max_j v_j^m$$

for $m \geq 0$.

(Continued on page 6.)

Løsningsforslag: If $0 < r \leq 1$ then v_j^{m+1} is a convex combination of $v_{j\pm 1}^m$ and therefore

$$\min_j v_j^m \leq \min \{v_{j-1}^m, v_{j+1}^m\} \leq v_j^{m+1} \leq \max \{v_{j-1}^m, v_{j+1}^m\} \leq \max_j v_j^m.$$

Assume from now on that r satisfies this condition.

4b

Assume that w_j^m solves

$$L_{\Delta x} w_j^m = g_j^m$$

for $m \geq 0$ and $j = 0, \dots, N$ with periodic boundary conditions $w_{-1}^m = w_N^m$, $w_{N+1}^m = w_0^m$. Here g_j^m is a given grid function. We assume that $w_j^0 = 0$ for all relevant j . Show that

$$\max_j |w_j^{m+1}| \leq (m+1)\Delta t \max_{\substack{j=0,\dots,N \\ k=0,\dots,m}} |g_j^k|.$$

Løsningsforslag: For $m = -1$ the estimate holds. Assume that it holds for $m - 1$, then

$$\begin{aligned} |w_j^{m+1}| &\leq \left| \frac{1}{2} (1-r) w_{j+1}^m + \frac{1}{2} (1+r) w_{j-1}^m \right| + \Delta t |g_j^m| \\ &\leq \max_{j=0,\dots,N} |w_j^m| + \Delta t |g_j^m| \\ &\leq m\Delta t \max_{\substack{j=0,\dots,N \\ k=0,\dots,m-1}} |g_j^k| + \max_{j=0,\dots,N} |g_j^m| \\ &\leq (m+1)\Delta t \max_{\substack{j=0,\dots,N \\ k=0,\dots,m}} |g_j^k|. \end{aligned}$$

4c

Let u be a smooth solution of (1), show that

$$L_{\Delta x} u(x_j, t^m) = \mathcal{O}(\Delta x),$$

and use this to obtain a bound of the error

$$\max_{j=0,\dots,N} |v_j^m - u(x_j, t^m)|.$$

(Continued on page 7.)

Løsningsforslag: We have that (with $u_j^m = u(x_j, t^m)$)

$$\begin{aligned} u_j^m &= \frac{1}{2} (u_{j+1}^m + u_{j-1}^m) + \mathcal{O}(\Delta x^2), \\ u_t(x_j, t^m) &= \frac{1}{\Delta t} (u_j^{m+1} - u_j^m) + \mathcal{O}(\Delta t) \\ &= \frac{1}{\Delta t} \left(u_j^{m+1} - \frac{1}{2} (u_{j+1}^m + u_{j-1}^m) + \mathcal{O}(\Delta x^2) \right) + \mathcal{O}(\Delta t), \\ u_x(x_j, t^m) &= \frac{1}{2\Delta x} (u_{j+1}^m - u_{j-1}^m) + \mathcal{O}(\Delta x^2). \end{aligned}$$

Since $\Delta t = r\Delta x$, $\mathcal{O}(\Delta x) = \mathcal{O}(\Delta t)$, using the above we find that

$$L_{\Delta x} u_j^m = u_t + u_x + \mathcal{O}(\Delta x).$$

Define $e_j^m = u_j^m - v_j^m$, we find that

$$\begin{cases} L_{\Delta x} e_j^m = \mathcal{O}(\Delta x), \\ e_j^0 = 0. \end{cases}$$

By **b)**, we get

$$\max_j |e_j^m| \leq t^m \mathcal{O}(\Delta x).$$

THE END