

MAT3360, vår 2022

Obligatorisk oppgave nr. 1 av 2.

Innleveringsfrist. Torsdag 3. mars 2022, klokken 14:30, i Canvas (canvas.uio.no).

Merk at man har **ett forsøk** på å få oppgaven godkjent. Dette betyr at det ikke lenger gis andregangsforsøk.

Du velger selv om du skriver besvarelsen for hånd og scanner besvarelsen eller om du skriver løsningen direkte inn på datamaskin (for eksempel ved bruk av $\text{\LaTeX}$). Besvarelsen skal leveres som én PDF-fil. Scannede ark må være godt lesbare. Besvarelsen skal inneholde navn, emne og oblignummer.

Det forventes at man har en klar og ryddig besvarelse med tydelige begrunnelser. Husk å inkludere alle relevante plott og figurer. Samarbeid og alle slags hjelpemidler er tillatt, men den innleverte besvarelsen skal være skrevet av deg og reflektere din forståelse av stoffet. Er vi i tvil om du virkelig har forstått det du har levert inn, kan vi be deg om en muntlig redegjørelse.

I oppgaver der du blir bedt om å programmere må du legge ved programkoden og levere den sammen med resten av besvarelsen. Det er viktig at programkoden du leverer inneholder et kjøreeksempel, slik at det er lett å se hvilket resultat programmet gir.

Søknad om utsettelse av innleveringsfrist. Hvis du blir syk eller av andre grunner trenger å søke om utsettelse av innleveringsfristen, må du ta kontakt med studieadministrasjonen ved Matematisk institutt (e-post: studieinfo@math.uio.no) senest samme dag som innleveringsfristen.

For å få adgang til avsluttende eksamen i dette emnet, må man bestå alle obligatoriske oppgaver i ett og samme semester.

For fullstendige retningslinjer for innlevering av obligatoriske oppgaver, se her:

www.uio.no/studier/admin/obligatoriske-aktiviteter/mn-math-oblig.html

LYKKE TIL!

You must have a score of at least 60% to pass the exercise.

Background. We say that a function w is piecewise differentiable on the interval $[0, 1]$ if w is continuous on $[0, 1]$ and $w'(x)$ exists for all $x \in [0, 1]$ except for a finite number of internal points $\{x_j\}$, $j = 1, \dots, M$, and the limits

$$\lim_{x \nearrow x_j} w'(x) \text{ and } \lim_{x \searrow x_j} w'(x)$$

both exist for all j (and are therefore finite). We let H denote the space of piecewise differentiable functions on $[0, 1]$.

The space $C_0^2([0, 1])$ is defined to consist of those functions w in $C^2((0, 1)) \cap C([0, 1])$ such that $w(0) = w(1) = 0$. We also use the notation

$$\|u\| = \left(\int_0^1 u^2(x) dx \right)^{1/2}$$

for the mean square norm. The Cauchy-Schwartz inequality

$$\int_0^1 u(x)v(x) dx \leq \|u\| \|v\|$$

will also be useful.

Consider the boundary value problem

$$(1) \quad \begin{cases} -u'' = f & x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

where f is a known function in $C((0, 1))$. We say that u is a *classical* solution of (1) if $u \in C_0^2([0, 1])$ and $-u''(x) = f(x)$ for all $x \in (0, 1)$.

Let

$$H_0 = \{u \in H \mid u(0) = u(1) = 0\}.$$

We say that u is a *weak* solution of (1) if $u \in H_0$ and

$$(2) \quad \int_0^1 u'(x)v'(x) dx = \int_0^1 f(x)v(x) dx \text{ for all } v \in H_0.$$

Question 1. Show that for each u in H there is a constant $L < \infty$ such that

$$|u(x) - u(y)| \leq L |x - y| \text{ for all } x, y \in [0, 1].$$

Possible solution: Observe that for each interval $[x_j, x_{j+1}]$ we have that $|u'(x)| \leq L_j < \infty$. Set $L = \max_j L_j$. Then

$$|u(x) - u(y)| \leq \left| \int_y^x u'(z) dz \right| \leq L |x - y|.$$

Question 2. Show that

$$|u(x)|^2 \leq \min \{x, 1-x\} \|u'\|^2 \quad \text{for } u \in H_0.$$

Possible solution: For $u \in H_0$, $u(x) = \int_0^x 1 u'(y) dy$. Using the Cauchy-Schwartz inequality gives

$$|u(x)| \leq \sqrt{x} \|u'\|.$$

From $u(x) = -\int_x^1 u'(y) dy$ we get the other estimate.

Question 3. Show that classical solutions are weak solutions.

Possible solution: Since $u \in C_0^2([0, 1])$, u is in H_0 . Let v be a function in H_0 , multiplying the equation with v and integrating over $[0, 1]$ we get

$$\begin{aligned} \int_0^1 f(x)v(x) dx &= \int_0^1 -u''(x)v(x) dx \\ &= -u'(x)v(x) \Big|_{x=0}^{x=1} + \int_0^1 u'(x)v'(x) dx. \end{aligned}$$

The boundary terms vanish since $v \in H_0$.

Question 4. Show that weak solutions are unique.

Possible solution: Set $w = u_1 - u_2$ where u_1 and u_2 are weak solutions, then $\int_0^1 w'v' dx = 0$ for all $v \in H_0$, in particular also for $v = w$. Thus $\int_0^1 (w')^2 dx = 0$. Hence $w'(x) = 0$ for all $x \in (0, 1)$ except $\{x_j\}$. This means that $w(x) = c$ for all $x \in [0, 1]$, but $w(0) = 0$ which implies $w = 0$.

Galerkin's method. Let the V_N be a subspace of H_0 spanned by the linearly independent functions $\varphi_k \in H_0$, $k = 1, \dots, N$. Define $u_N \in V_N$ (which means that $u_N = \sum_{k=1}^N c_k \varphi_k$) to be the solution of

$$(3) \quad \int_0^1 u_N'(x)v'(x) dx = \int_0^1 f(x)v(x) dx \quad \text{for all } v \in V_N.$$

Question 5. Let u be the (if it exists) weak solution of (1), show that

$$\int_0^1 (u - u_N)'(x)v'(x) dx = 0 \quad \text{for all } v \in V_N.$$

Possible solution: Subtract (3) from (2) with $v \in V_N \subset H_0$.

Question 6. Show that

$$\|(u - u_N)'\| \leq \inf_{v \in V_N} \|(u - v)'\| \quad (\text{Cea's inequality}).$$

Possible solution: Let $v \in V_N$ and compute

$$\begin{aligned} \|(u - u_N)'\|^2 &= \int_0^1 (u' - u_N')^2 dx \\ &= \int_0^1 (u' - u_N')(u' - v') dx - \underbrace{\int_0^1 (u' - u_N')(u_N' - v') dx}_{=0} \\ &\leq \|(u - u_N)'\| \|u' - v'\|. \end{aligned}$$

Since v is arbitrary the result follows.

Question 7. Show that

$$|u(x) - u_N(x)|^2 \leq \min\{x, 1 - x\} \left(\inf_{v \in V_N} \|(u - v)'\| \right)^2.$$

Possible solution: This follows from 2 and 6.

Question 8. Let $u_N = \sum_{k=1}^N c_k \varphi_k$ and assume that u_N satisfies (3). We write $\mathbf{c} = (c_1, \dots, c_N)^T$, find an $N \times N$ symmetric matrix $\mathbf{A}$ and a vector $\mathbf{b}$ such that $\mathbf{A}\mathbf{c} = \mathbf{b}$, i.e., find the components a_{kl} and b_k given $\{\varphi_k\}$ and f .

Possible solution: We use $v = \varphi_l$ in (3) for $l = 1, \dots, N$ to get

$$\sum_{k=1}^N \int_0^1 \varphi_l' \varphi_k' dx c_k = \int_0^1 f \varphi_l dx, \quad \text{for } l = 1, \dots, N.$$

Therefore

$$a_{lk} = \int_0^1 \varphi_l' \varphi_k' dx \quad \text{and} \quad b_l = \int_0^1 f \varphi_l dx.$$

Question 9. Find a formula for u_N (involving f) if $\varphi_l(x) = \sin(l\pi x)$.

Possible solution: In this case

$$\begin{aligned} a_{kl} &= \int_0^1 kl\pi^2 \cos(k\pi x) \cos(l\pi x) dx \\ &= \begin{cases} 0 & k \neq l, \\ \frac{1}{2}k^2\pi^2 & k = l. \end{cases} \end{aligned}$$

Thus $\mathbf{A}$ is diagonal, and we get

$$c_k = \frac{2}{(k\pi)^2} \int_0^1 f(x) \sin(k\pi x) dx.$$

Now define $h = 1/(N + 1)$, $x_k = kh$, and set

$$\psi(x) = \max \{1 - |x|, 0\}.$$

For $k = 1, \dots, N$ set

$$\varphi_k(x) = \psi\left(\frac{x - x_k}{h}\right).$$

From now on we shall let V_N be the space spanned by these functions. (In order to get the idea of what these functions look like, you can draw the graphs of φ_1 , φ_2 and φ_3 if $N = 3$.)

Question 10. Assume that there exists a classical solution to (1), and let u_N solve (3). Show the error estimate

$$(4) \quad |u(x) - u_N(x)| \leq \frac{1}{\sqrt{2}} \sup_{x \in (0,1)} |f(x)| h$$

for all $x \in [0, 1]$.

Possible solution: Let $v \in V_N$ be the linear interpolation of u at the points x_k , i.e.,

$$v(x) = \sum_{k=1}^N u(x_k) \varphi_k(x).$$

For $x \in (x_k, x_{k+1})$ we have that

$$v'(x) = \frac{1}{h} (u(x_{k+1}) - u(x_k)) = u'(x_{k+1/2})$$

for some intermediate value $x_{k+1/2}$. Hence

$$|u'(x) - v'(x)| \leq \max |u''| h.$$

We get that

$$\begin{aligned}\|(u - v)'\|^2 &= \sum_{k=0}^N \int_{x_k}^{x_{k+1}} (u' - v')^2 dx \\ &\leq \sum_{k=0}^N \int_{x_k}^{x_{k+1}} \left(\sup_{x \in (0,1)} |f(x)| \right)^2 h^2 dx \\ &= \left(\sup_{x \in (0,1)} |f(x)| \right)^2 h^2.\end{aligned}$$

Question 7 implies that the error estimate holds.

Question 11. Find the matrix $\mathbf{A}$, then find the vector $\mathbf{b}$ if $f(x) = \pi^2 \sin(\pi x)$.

Possible solution: We have that

$$\varphi'_k(x) = \begin{cases} \frac{1}{h} & x \in (x_{k-1}, x_k) \\ -\frac{1}{h} & x \in (x_k, x_{k+1}) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$a_{kl} = \begin{cases} 0 & l \notin \{k-1, k, k+1\} \\ -\frac{1}{h} & l \in \{k-1, k+1\} \\ \frac{2}{h} & l = k. \end{cases}$$

Thus $\mathbf{A}$ is tridiagonal

$$\mathbf{A} = \frac{1}{h} \begin{pmatrix} 2 & -1 & 0 & & 0 & 0 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -1 & 2 & -1 \\ 0 & 0 & & 0 & -1 & 2 \end{pmatrix}$$

$$\begin{aligned}
b_k &= \int_0^1 \pi^2 \sin(\pi x) \varphi_k(x) dx \\
&= \pi \int_0^1 \cos(\pi x) \varphi'_k(x) dx \\
&= \pi \frac{1}{h} \left(\int_{x_{k-1}}^{x_k} \cos(\pi x) dx - \int_{x_k}^{x_{k+1}} \cos(\pi x) dx \right) \\
&= \frac{1}{h} (2 \sin(\pi x_k) - \sin(\pi x_{k+1}) - \sin(\pi x_{k-1})) \\
&= \frac{2}{h} \sin(\pi x_k) (1 - \cos(\pi h)) \\
&= \frac{4}{h} \sin^2(\pi h/2) \sin(\pi x_k).
\end{aligned}$$

(Note that if $f(x) = \pi^2 \sin(\pi x)$ the classical solution is $u(x) = \sin(\pi x)$.)

Question 12. Implement the method on a computer (that is, define $\mathbf{A}$ and $\mathbf{b}$, then solve the equation $\mathbf{A}\mathbf{c} = \mathbf{b}$, you don't have to program the actual solver, use “\” in Matlab, similar software exists in Python, Julia etc.). Comment your code!

Test your code for $N = 10$. What are the errors?

Possible solution: Here is a Matlab code

```

N=10;
e=ones(N,1); % Building the matrix A (look up "spdiags")
A=spdiags([-e 2*e -e],[-1:1,N,N]);

h=1/(N+1); % Setting the x coordinate
x=(h:h:1-h)';

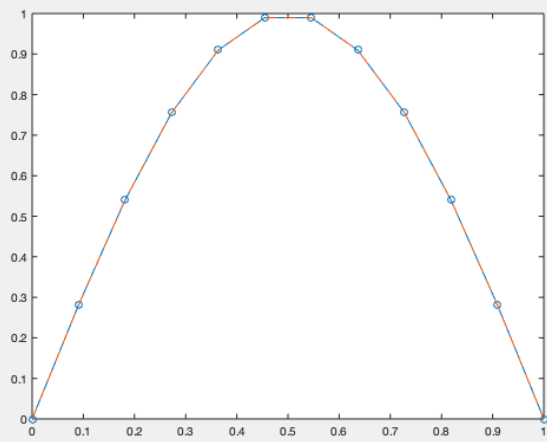
hh=(sin(pi*h/2)).^2; % Calculating the right hand side b
b=4*hh*sin(pi*x); % (I have multiplied with h)

u=A\b; % Solving the equation

u=[0;u;0]; x=[0;x;1]; % Adding boundary and boundary values
exact=sin(pi*x);
error=abs(u-exact); % Calculating the error
max(error)
plot(x,u,'o-',x,exact,'--'); % Plotting the result

```

Here is a plot of the approximate and exact solution for $N = 10$.



The errors are very small.