

Mat 3360 Lecture 1

Differential equation (DE): equation that relates an unknown function to some of its derivatives.

Examples

- $u'(t) = u(t)$ (ODE)
- $(u''(x))^2 + \sin(x) u'(x) = 1 - \cos(x)$ (ODE)
- $u_t(x,t) + \cos(x) u_x(x,t) = x^2$ (PDE)
- $u_{xx}(x,y) + u_{yy}(x,y) = e^{-x^2}$ (PDE)

When unknown function is multivariate, the DE is called a partial differential equation (PDE)

Otherwise (when unknown is univariate), it is an ordinary differential equation (ODE)

Notation for partial derivatives of $u(x,t)$:

$$u_t = \frac{\partial u}{\partial t}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2} \quad \text{etc.}$$

Terminology

The order of a DE refers to the highest derivative in equation.

EX

- $u''(x) + e^{u(x)} = \sin(x)$

is second order

- $u_{ttx}(x,t) + (u_{tt}(x,t))^3 = \sin(t)$

is third order

A DE on the form

$$\underbrace{\mathcal{L}(u)}_{\text{(all terms involving unknown)}} = \underbrace{f}_{\text{(remaining terms)}} \quad (1)$$

is called homogeneous if $f \equiv 0$,
otherwise nonhomogeneous
(i.e., when $f \neq 0$).

$\mathcal{L}(u) = 0$ is called the homogeneous part of (1).

Ex

$$u_t(x,t) = e^t u_{xx}(x,t) + \sin(xt)$$

$$\Rightarrow \underbrace{u_t(x,t) - e^t u_{xx}(x,t)}_{\mathcal{L}(u)} = \underbrace{\sin(xt)}_{=f(x,t) \neq 0}$$

is nonhomogeneous.

A DE (1) is called linear if its homogeneous part $\mathcal{L}(u) = 0$ satisfies

$$\mathcal{L}(\alpha u + \beta v) = \alpha \mathcal{L}(u) + \beta \mathcal{L}(v) \quad (2)$$

for all $\alpha, \beta \in \mathbb{R}$ and all suitable functions u, v .

If (2) does not hold, the DE is nonlinear.

Ex

given

$$\underbrace{u_t(x,t) + a(x,t) u_x}_{\mathcal{L}(u)} = 0$$

$\mathcal{L}(u)$

is homogeneous and linear since

$$\begin{aligned}\mathcal{L}(\alpha u + \beta v) &= (\alpha u + \beta v)_t + a(x,t)(\alpha u + \beta v)_x \\ &= \alpha u_t + \beta v_t + a(x,t)(\alpha u_x + \beta v_x) \\ &= \alpha(u_t + a(x,t)u_x) + \beta(v_t + a(x,t)v_x) \\ &= \alpha \mathcal{L}(u) + \beta \mathcal{L}(v)\end{aligned}$$

(Nonlinear terms in u , like u^2 , e^{u_x} , $u_t u_x$, etc. make $\mathcal{L}(u)$ nonlinear.)

First numerical method

Consider ODE

$$\left. \begin{array}{l} u'(t) = u(t) \\ u(0) = u_0 \end{array} \right\} \Rightarrow u(t) = u_0 e^t$$

Approximation

$$u'(t) \approx \frac{u(t+\Delta t) - u(t)}{\Delta t}$$

leads to equation

$$\left. \begin{array}{l} \frac{V(t+\Delta t) - V(t)}{\Delta t} = V(t) \\ V(0) = u_0 \end{array} \right\} (3)$$

Solve on mesh $t_m = m\Delta t, m=0,1,\dots$

Then

$$\begin{aligned} V(t_{m+1}) &= V(t_m) + \Delta t V(t_m) \\ &= (1 + \Delta t) V(t_m) \\ &= (1 + \Delta t)^{m+1} u_0 \end{aligned}$$

For $t > 0$, take Δt s.t. $M = \frac{t}{\Delta t}$ is an integer. Then $t_M = t$ and

$$|v(t_M) - u(t_M)| = |(1 + \Delta t)^M u_0 - e^t u_0|$$

$$= |u_0| \underbrace{\left| (1 + \Delta t)^{\frac{t}{\Delta t}} - e^t \right|}$$

$\rightarrow 0$ as $\Delta t \rightarrow 0$

$\Rightarrow$ numerical method converges.

Transport equations

Want to solve (first-order, linear homogeneous) the PDE

$$\left. \begin{aligned} u_t(x, t) + a(x, t)u_x(x, t) &= 0 & x \in \mathbb{R}, t > 0 \\ \text{with } u(x, 0) &= \varphi(x) & x \in \mathbb{R} \end{aligned} \right\} (4)$$

where a and φ are given smooth functions.

Consider the ODE

$$\left. \begin{aligned} x'(t) &= a(x(t), t) \\ x(0) &= x_0 \end{aligned} \right\} t > 0 \quad (5)$$

and the function

$$h(t) := u(x(t), t) \quad \text{where } u \text{ solves (4).}$$

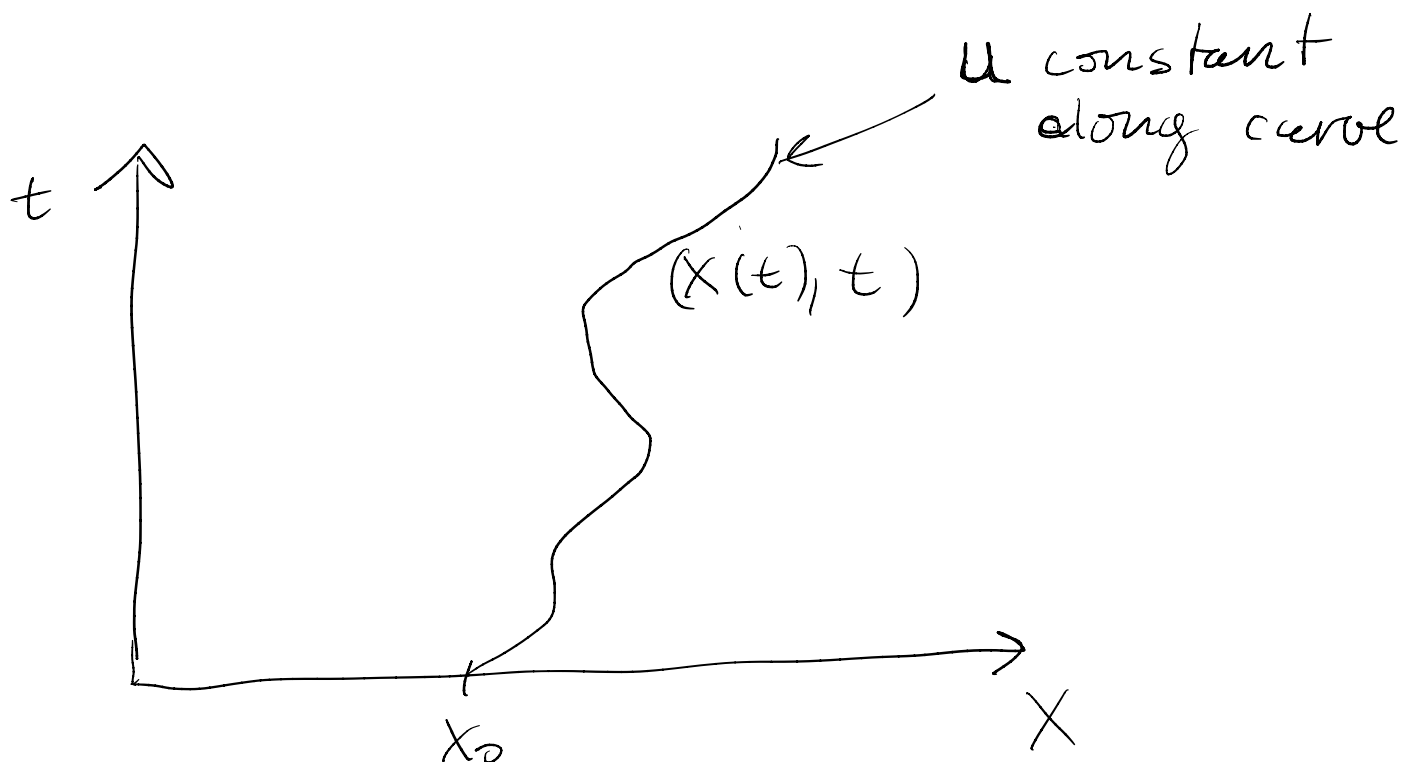
Then h is constant, as

$$\begin{aligned}h'(t) &= \frac{d}{dt} u(x(t), t) \\&= u_x(x(t), t) x'(t) + u_t(x(t), t) \\&= u_t(x(t), t) + a(x(t), t) u_x(x(t), t) \\&\stackrel{(4)}{=} 0\end{aligned}$$

$$\Rightarrow h(t) = h(0)$$

$$\Rightarrow u(x(t), t) = u(x_0, 0) = q(x_0) \quad (6)$$

u is constant along characteristic curves $(x(t), t)$



Method of characteristics:

1. Find inversion mapping

$(x, t) \mapsto x_0(x, t)$ s.t. ODE (5)
satisfies $x(t) = x$

2. Formal solution

$$u(x, t) \stackrel{(6)}{=} \varphi(x_0(x, t))$$

Examples

$a(x, t) = a$ constant in

$$u_t + a u_x = 0$$

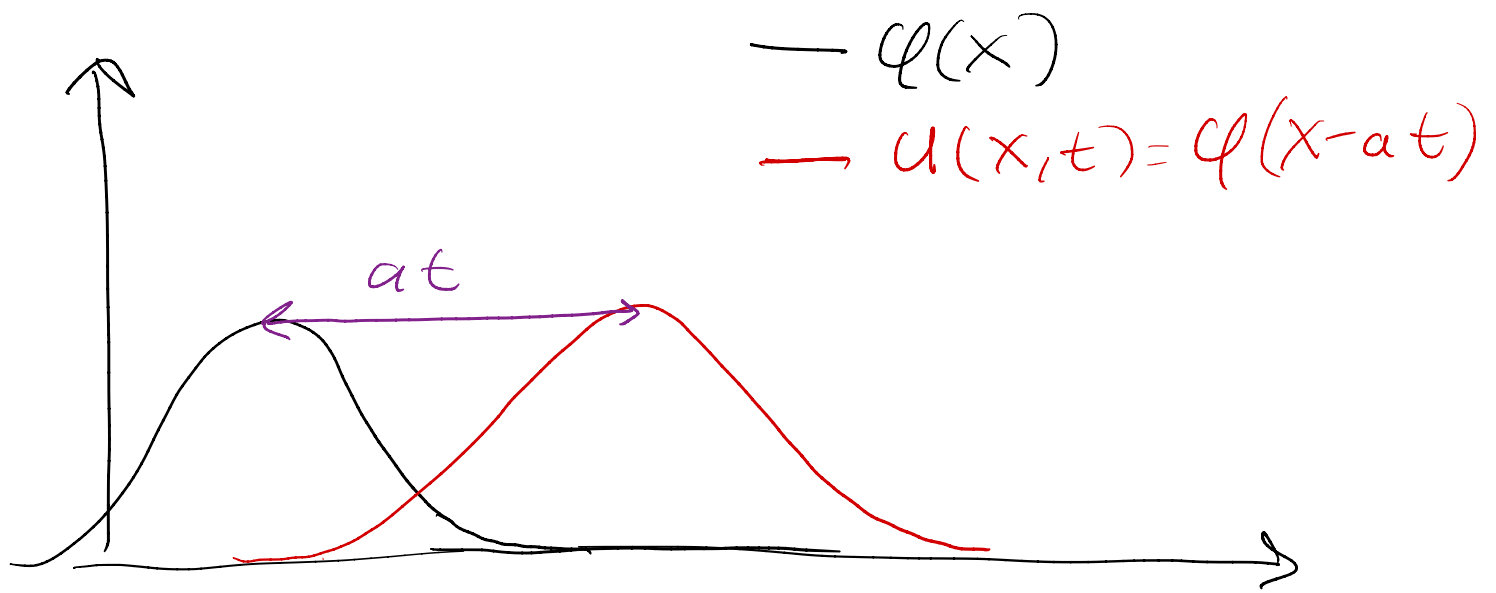
$$u(x, 0) = \varphi(x)$$

ODE

$$\left. \begin{array}{l} x'(t) = a \\ x(0) = x_0 \end{array} \right\} \Rightarrow x(t) = x_0 + a t$$

Inversion: $x_0 = x - a t$

$$\text{and } u(x, t) = \varphi(x_0(x, t)) = \varphi(x - a t)$$



(solution transported along x -axis)

Example

$$u_t + x u_x = 0$$

$$u(x, 0) = \sin(x)$$

ODE

$$X'(t) = X(t)$$

$$X(0) = X_0$$

$$\Rightarrow X(t) = X_0 e^t$$

$$\Rightarrow X_0(x, t) = e^{-t} x$$

$$\text{and } u(x, t) = u(X_0(x, t), 0)$$

$$= \sin(e^{-t} x)$$

Nonhomogeneous PDE

consider next

$$u_t + a(x, t) u_x = b(x, t)$$

$$u(x, 0) = \varphi(x)$$

with a, b, φ given smooth

Equation for characteristics:

$$x'(t) = a(x(t), t)$$

$$x(0) = x_0.$$

Study again $h(t) = u(x(t), t) :$

$$h'(t) = \frac{d}{dt} u(x(t), t)$$

$$= u_t(x(t), t) + a(x(t), t) u_x(x(t), t)$$

$$= b(x(t), t)$$

$$\Rightarrow h(t) = h(0) + \int_0^t b(x(s), s) ds.$$

Solution:

$$u(x, t) = \varphi(x_0(x, t)) + \int_0^t b(x(s), s) ds$$

Example

$$u_t + u_x = \cos(x)$$

$$u(x, 0) = \varphi(x)$$

ODE

$$\left. \begin{array}{l} X'(t) = 1 \\ X(0) = x_0 \end{array} \right\} \Rightarrow X(t) = x_0 + t$$

$$\& x_0 = x - t$$

$$u(x, t) = \varphi(x_0) + \int_0^t \cos(X(s)) ds$$

$$= \varphi(x_0) + \sin(x_0 + t) - \sin(x_0)$$

$$= \varphi(x - t) + \sin(x) - \sin(x - t)$$