

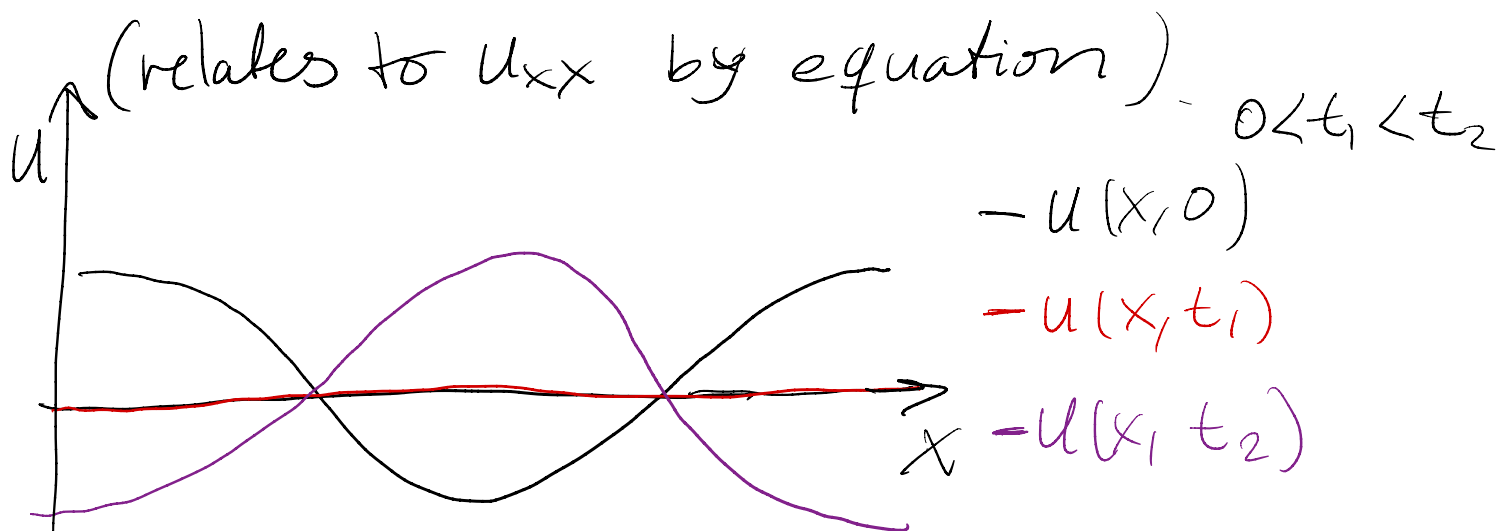
MAT 3360 Lecture 2

The wave equation

$$\left. \begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) & t > 0, x \in \mathbb{R} \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned} \right\} (1)$$

models for example vibrating string

- $u(x,t)$ displacement from stationary state $u \equiv 0$ at point x and time t
- $u_t(x,t)$ velocity of displacement at (x,t)
- $u_{tt}(x,t)$ acceleration ——— || ———



Solve (i) by introducing

$y = x + t$ and $z = x - t$,
and define

$$v(y, z) = u(x, t)$$

(given by $v(y, z) = u\left(\frac{y+z}{2}, \frac{y-z}{2}\right)$)

Chain rule gives

$$u_x = v_y \frac{\partial y}{\partial x} + v_z \frac{\partial z}{\partial x} = v_y + v_z$$

$$u_{xx} = v_{yy} + 2v_{yz} + v_{zz}$$

and similarly

$$u_t = v_y - v_z \text{ and}$$

$$u_{tt} = v_{yy} - 2v_{yz} + v_{zz}.$$

Wave eq. yields that

$$0 = u_{tt} - u_{xx} = -4v_{yz}$$

$$\Rightarrow v_{yz} = 0$$

$$\Rightarrow v(y, z) = f(y) + g(z)$$

for some functions f and g .
(Show this in exercise 1.12).

This gives

$$\left. \begin{aligned} u(x,t) = v(y,z) &= f(y) + g(z) \\ &= f(x+t) + g(x-t) \end{aligned} \right\} (2)$$

Determine f & g by initial condition in (1):

$$\varphi(x) = u(x,0) = f(x) + g(x) \quad (3)$$

$$\Rightarrow \varphi'(x) = f'(x) + g'(x) \quad (4)$$

$$\text{And } u_t(x,t) = f'(x+t) - g'(x-t)$$

$$\Rightarrow \psi(x) = u_t(x,0) = f'(x) - g'(x) \quad (5)$$

(4) and (5) gives us

$$f' = \frac{\varphi' + \psi}{2} \quad \text{and} \quad g' = \frac{\varphi' - \psi}{2}$$

By integration,

$$\left. \begin{aligned} f(s) &= \frac{\varphi(s)}{2} + C_1 + \frac{1}{2} \int_0^s \psi(\theta) d\theta \\ g(s) &= \frac{\varphi(s)}{2} + C_2 - \frac{1}{2} \int_0^s \psi(\theta) d\theta \end{aligned} \right\} (6)$$

Consequently,

$$\left. \begin{aligned} f(s) + g(s) &= \varphi(s) + c_1 + c_2, \\ \text{but (3) tells us that} \\ f(s) + g(s) &= \varphi(s) \end{aligned} \right\} \Rightarrow c_1 + c_2 = 0.$$

Obtain solution from (2) and (6):

$$u(x, t) = f(x+t) + g(x-t)$$

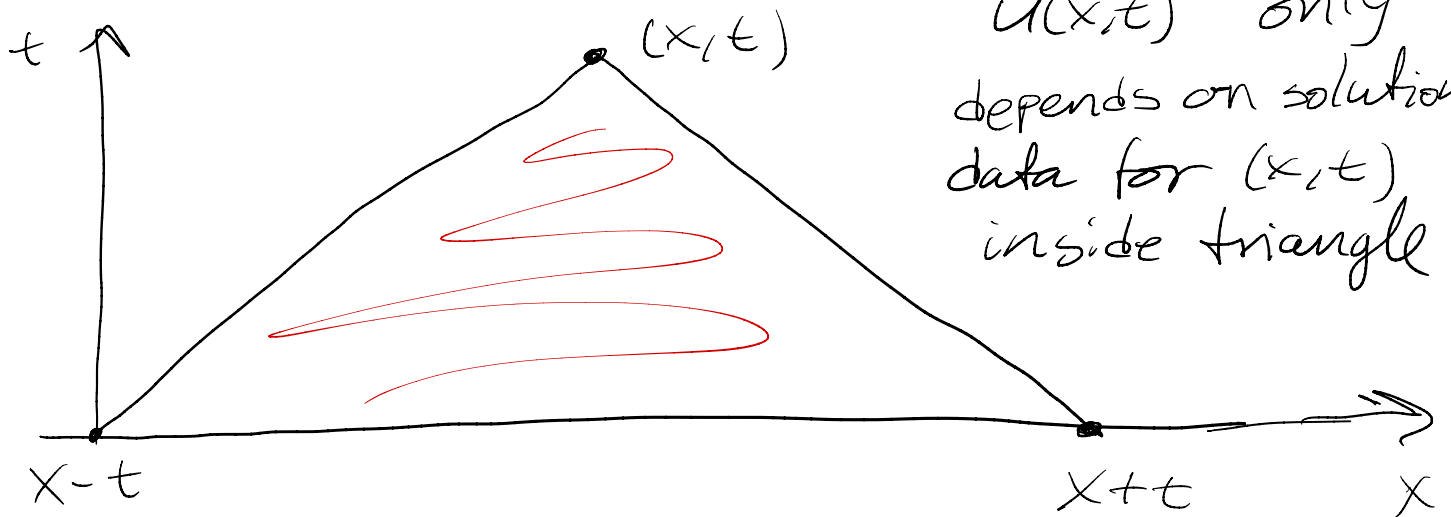
$$= \frac{1}{2} \left[\varphi(x+t) + \varphi(x-t) + \int_0^{x+t} \psi(\theta) d\theta - \int_0^{x-t} \psi(\theta) d\theta \right]$$

$$= \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(\theta) d\theta$$

d'Alembert's formula

$$u(x,t) = \frac{\phi(x+t) + \phi(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \psi(\theta) d\theta$$

Domain of dependence:



Example

$$u_{tt} = u_{xx}$$

$$\varphi(x) = \sin(x)$$

$$\psi(x) = \cos(x)$$

d'Alembert's formula:

$$u(x,t) = \frac{\sin(x+t) + \sin(x-t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} \cos(\theta) d\theta$$

$$= \frac{\sin(x+t) + \sin(x-t)}{2} + \frac{\sin(x+t) - \sin(x-t)}{2}$$

$$= \sin(x+t)$$

Heat equation

$$\left. \begin{aligned} u_t(x,t) &= u_{xx}(x,t) & x \in \mathbb{R}, t > 0 \\ u(x,0) &= H(x) := \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \end{aligned} \right\} (7)$$

Can model evolution of temperature in homogeneous media where

$u(x,t)$ — temperature at point x at time t

$u_t(x,t)$ — change in temperature at (x,t)
(relates to u_{xx} by eq.)

Steps solution of (7):

1 show that $u(x, t) = w\left(\frac{x}{\sqrt{t}}\right) \quad \forall t > 0$
for some $w: \mathbb{R} \rightarrow \mathbb{R}$.

2. Insert w in (7) to obtain ODE
for w .

3. Solve ODE to determine w and u .

Step 1.

For any $c > 0$, define

$$v(x, t) = u(cx, c^2 t).$$

Then $v(x, 0) = u(cx, 0) = H(cx) = H(x)$

and $v_t(x, t) = c^2 u_t(cx, c^2 t)$

$$v_{xx}(x, t) = c^2 u_{xx}(cx, c^2 t)$$

This and (7) implies that

$$v_t(x, t) = v_{xx}(x, t)$$

Observation: For any $c > 0$,

$v(x, t) = u(cx, c^2t)$ is solution of (7)
& uniqueness of solution (TAKE AS FACT)

$$\Rightarrow u(x, t) = u(cx, c^2t) \quad \forall c > 0$$

Taking $c = 1/\sqrt{t}$ gives

$$u(x, t) = u\left(\frac{x}{\sqrt{t}}, 1\right).$$

This means the solution must be on the form

$$u(x, t) = W\left(\frac{x}{\sqrt{t}}\right) \quad \text{for some } W: \mathbb{R} \rightarrow \mathbb{R}.$$

Step 2. Introduce variable $\gamma = \frac{x}{\sqrt{t}}$ and determine $W(\gamma)$.

Boundary conditions:

For fixed $x < 0$, we formally have

$$\begin{aligned} W(-\infty) &= \lim_{t \downarrow 0} W\left(\frac{x}{\sqrt{t}}\right) = \lim_{t \downarrow 0} u(x, t) \\ &= H(x) = 0 \end{aligned}$$

And for $x > 0$,

$$W(\infty) = \lim_{t \downarrow 0} W\left(\frac{x}{\sqrt{t}}\right) = H(x) = 1$$

ODE for w :

By chain rule

$$u_t(x,t) = \frac{\partial}{\partial t} W(\gamma) = W'(\gamma) \frac{\partial \gamma}{\partial t} \\ = -t^{-1}(\gamma/2) W'(\gamma)$$

and

$$u_{xx}(x,t) = t^{-1} W''(\gamma)$$

Since $u_t - u_{xx} = 0$,

we obtain ODE

$$W''(\gamma) + (\gamma/2) W'(\gamma) = 0 \quad \gamma \in \mathbb{R}$$

$$w(-\infty) = 0, \quad w(\infty) = 1$$

Step 3. Multiply ODE with $e^{\gamma^2/4}$:

$$\Rightarrow (e^{\gamma^2/4} W'(\gamma))' = 0$$

$$\Rightarrow W'(\gamma) = C e^{-\gamma^2/4}$$

$$\Rightarrow w(\gamma) = c \int_{-\infty}^{\gamma} e^{-s^2/4} ds + \underbrace{w(-\infty)}_{=0}$$

$$= 2c \int_{-\infty}^{\gamma/2} e^{-\theta^2} d\theta$$

and determine constant by

$$1 = w(\infty) = 2c \int_{-\infty}^{\gamma/2} e^{-\theta^2} d\theta$$

$$= 2c \sqrt{\pi}$$

$$\Rightarrow c = \frac{1}{2\sqrt{\pi}}$$

so

$$w(\gamma) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\gamma/2} e^{-\theta^2} d\theta$$

and we obtain solution to (7):

$$u(x,t) = w\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x/2\sqrt{t}} e^{-\theta^2} d\theta$$

$u(x, 0)$
 $u(x, t)$

