

MAT 3360 Lecture 3

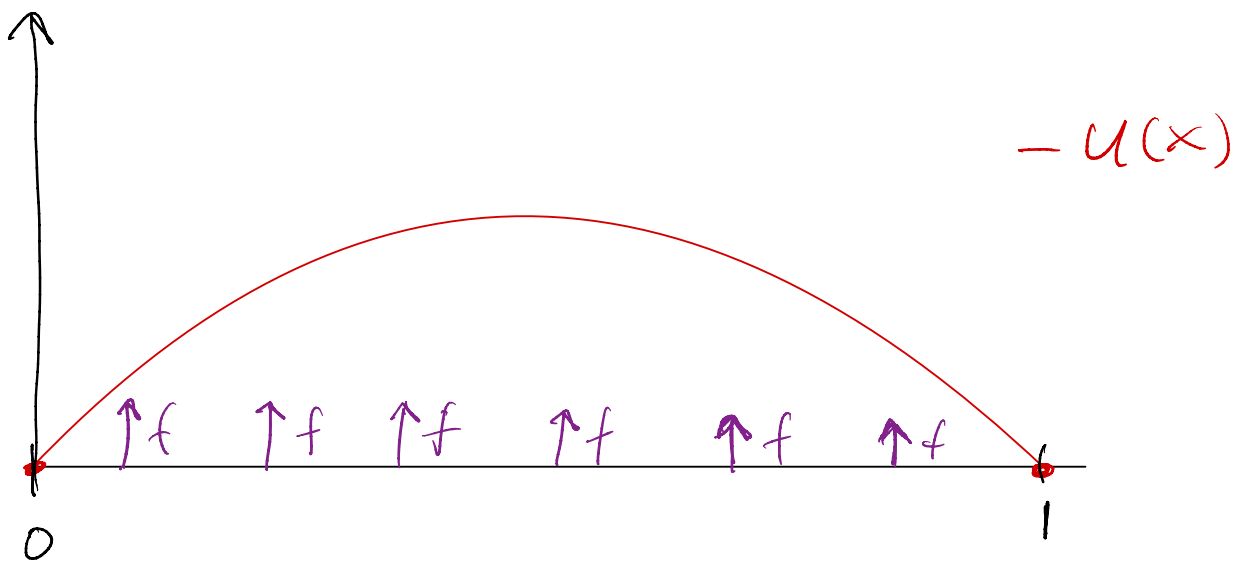
Poisson's equation in 1D

$$\left. \begin{aligned} -u''(x) &= f(x) \\ u(0) &= u(1) = 0 \end{aligned} \right\} x \in (0,1) \quad (1)$$

can model

$u(x)$ - displacement of elastic string at x

$f(x)$ - force on string in y -direction



(can also model the electric potential u subject to charge density f).

Plan

1. Existence & uniqueness of solution to (1)
2. Representation of solutions using a Green's function.

1. Integration of (1):

$$u'(y) = c_1 + \int_0^y \underbrace{u''(z)}_{=-f(z)} dz = c_1 - \int_0^y f(z) dz$$

and

$$\begin{aligned} u(x) &= c_1 x + c_2 - \int_0^x \int_0^y f(z) dz dy \\ &= c_1 x + c_2 - \int_0^x F(y) dy \end{aligned}$$

$$\text{where } F(y) := \int_0^y f(z) dz.$$

Integration by parts:

$$\begin{aligned} \int_0^x F(y) dy &= [yF(y)]_0^x - \int_0^x y F'(y) dy \\ &= xF(x) - \int_0^x y f(y) dy \\ &= \int_0^x (x-y) f(y) dy \end{aligned}$$

$$\Rightarrow u(x) = C_2 + C_1 x - \int_0^x (x-y) f(y) dy$$

$$u(0) = 0 \Rightarrow C_2 = 0.$$

$$u(1) = 0 \Rightarrow C_1 = \int_0^1 (1-y) f(y) dy$$

Solution:

$$u(x) = x \int_0^1 (1-y) f(y) dy - \int_0^x (x-y) f(y) dy \quad (2)$$

Theorem For any $f \in C((0,1))$,
 the ODE $-u''(x) = f \quad x \in (0,1)$
 $u(0) = u(1) = 0$

has a unique solution given
 by (2), and $u \in C_0^2((0,1))$

Notation

$$C^m((0,1)) := \{g: (0,1) \rightarrow \mathbb{R} \mid g^{(m)} \in C((0,1))\}$$

$$C([0,1]) := \left\{ g: [0,1] \rightarrow \mathbb{R} \mid \text{such that} \right. \\ \left. g \in C((0,1)) \text{ and limits} \right. \\ \left. \lim_{x \downarrow 0} g(x) \text{ and } \lim_{x \uparrow 1} g(x) \text{ exist} \right\}$$

$$C_0^m((0,1)) := \{g \in C^m((0,1)) \cap C([0,1]) \mid g(0) = g(1) = 0\}$$

Improved regularity of solution:

If $f \in C^m((0,1))$, then $u \in C_0^{m+2}((0,1))$.

Examples

- For $f(x) \equiv 1$,

$$u(x) = x \int_0^1 (1-y) dy - \int_0^x (x-y) dy$$

$$= \frac{x(1-x)}{2}$$

(Note $u \in C_0^m((0,1))$ for any $m \geq 0$.)

• For $f(x) = x$,

$$\begin{aligned} u(x) &= x \int_0^1 (1-y) y \, dy - \int_0^x (x-y) y \, dy \\ &= \frac{x(1-x^2)}{6} \end{aligned}$$

Alternative solution representation

We rewrite (2)

$$\begin{aligned} u(x) &= \int_0^1 x(1-y) f(y) \, dy - \int_0^x (x-y) f(y) \, dy \\ &= \int_x^1 x(1-y) f(y) \, dy \\ &\quad + \int_0^x \underbrace{(x(1-y) - (x-y))}_{= y - xy} f(y) \, dy \end{aligned}$$

$$= \int_0^x y(1-x) f(y) \, dy + \int_x^1 x(1-y) f(y) \, dy$$

Introduce Green's function

$$G(x, y) := \begin{cases} y(1-x) & 0 \leq y \leq x \leq 1 \\ x(1-y) & 0 \leq x \leq y \leq 1 \end{cases}$$

Then

$$u(x) = \int_0^1 G(x, y) f(y) dy \quad (4)$$

Properties of G

- G is continuous on $[0, 1] \times [0, 1]$

- Symmetric: $G(x, y) = G(y, x)$

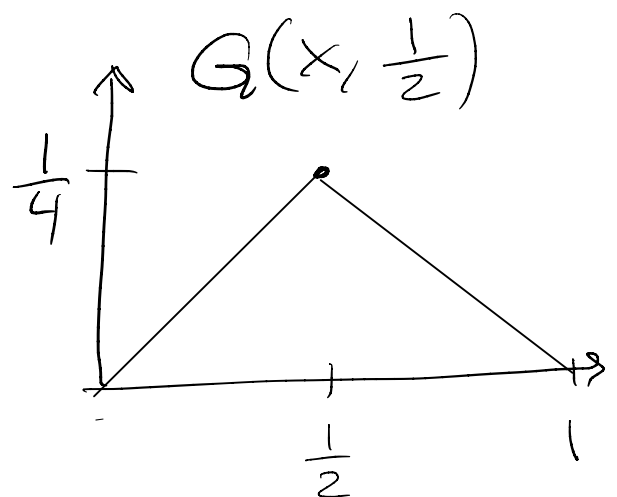
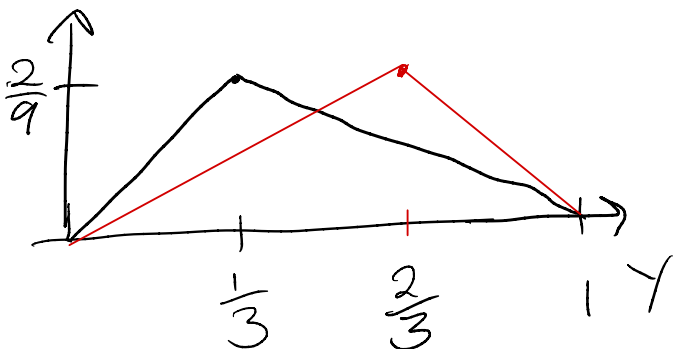
- Zero on boundary

$$G(x, 0) = G(x, 1) = G(0, y) = G(1, y) = 0$$

- $G(x, y) \geq 0$ for all $x, y \in [0, 1]$.

- For fixed x , $G(x, y)$ is piecewise linear function of y , and vice versa

$G(\frac{1}{3}, y)$, $G(\frac{2}{3}, y)$



For any $g \in C([0,1])$, we introduce the sup norm

$$\|g\|_{\infty} := \sup_{x \in [0,1]} |g(x)|$$

Theorem Let $f \in C([0,1])$ and let u be the unique solution of

$$\begin{aligned} -u''(x) &= f(x) & x \in (0,1) \\ u(0) &= u(1) = 0, \end{aligned}$$

$$\text{Then } \|u\|_{\infty} \leq \frac{1}{8} \|f\|_{\infty}$$

Proof By (4) and $G \geq 0$,

$$\begin{aligned} |u(x)| &= \left| \int_0^1 G(x,y) f(y) dy \right| \\ &\leq \int_0^1 G(x,y) |f(y)| dy \\ &\leq \|f\|_{\infty} \underbrace{\int_0^1 G(x,y) dy}_{\text{solution of (1) when } f \equiv 1} \\ &= \|f\|_{\infty} \frac{x(1-x)}{2}. \end{aligned}$$

$$\Rightarrow \sup_{x \in [0,1]} |u(x)| \leq \frac{\|f\|_{\infty}}{8}$$



Remarks

- If $f \geq 0$, then $u \geq 0$, since

$$u(x) = \int_0^1 \underbrace{G(x, \gamma)}_{\geq 0} \underbrace{f(\gamma)}_{\geq 0} d\gamma \geq 0.$$

(Non-negative force leads to non-negative solution.)

- Solutions to (1) may exist even if $f \notin C([0,1])$.

Example For

$$f(x) = H(x - \tfrac{1}{2}) = \begin{cases} 0 & x \leq \frac{1}{2} \\ 1 & x > \frac{1}{2} \end{cases}$$

By (2),

$$\begin{aligned} u(x) &= x \int_0^1 (1-y) H(y - \tfrac{1}{2}) dy - \int_0^x (x-y) H(y - \tfrac{1}{2}) dy \\ &= x \int_{\frac{1}{2}}^1 (1-y) dy - \int_{\frac{1}{2}}^{\max(x, \frac{1}{2})} (x-y) dy \end{aligned}$$

$$= x \left[-\frac{(1-y)^2}{2} \right]_{\frac{1}{2}}^1 + \left[\frac{(x-y)^2}{2} \right]_{\frac{1}{2}}^{\max(x, \frac{1}{2})}$$

$$= \begin{cases} \frac{x}{8} & x \leq \frac{1}{2} \\ \frac{x}{8} - \frac{(x - 1/2)^2}{2} & x > \frac{1}{2} \end{cases}$$

Here $u(0) = u(1) = 0$ and $u \in C_0'((0,1))$,
but $u \notin C_0^2((0,1))$ (since $f \notin C((0,1))$).

Numerical method

$$\left. \begin{aligned} -u''(x) &= f(x) & x \in (0,1) \\ u(0) &= u(1) = 0 \end{aligned} \right\} (5)$$

where we assume $f \in C^2([0,1])$.

Strategy: Approximate $u''(x)$ by finite differences and solve eqn on discrete mesh on $[0,1]$.

Taylor series

For $g \in C^4([0,1])$ and any points $0 \leq x < x+h \leq 1$,
$$g(x+h) = g(x) + g'(x)h + g''(x)\frac{h^2}{2} + g^{(3)}(x)\frac{h^3}{6} + \underbrace{g^{(4)}(\gamma)\frac{h^4}{4!}}_{=O(h^4)}$$
for some $\gamma \in [x, x+h]$.

Similarly, for any $0 \leq x-h < x \leq 1$,

$$g(x-h) = g(x) - g'(x)h + g''(x)\frac{h^2}{2} - g^{(3)}(x)\frac{h^3}{6} + O(h^4)$$

It follows that

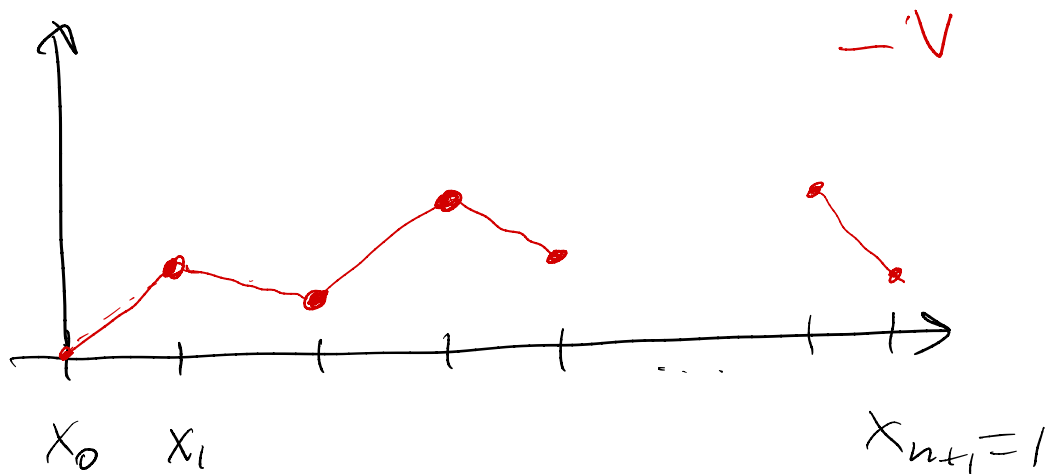
$$\frac{g(x+h) - 2g(x) + g(x-h)}{h^2} = g''(x) + O(h^2)$$

Mesh of $[0, 1]$:

Choose integer n , stepsize $h = \frac{1}{n+1}$
and meshpoints

$$x_j = jh \quad j = 0, 1, \dots, n+1$$

A sequence $V_j = V(x_j) \quad j = 0, 1, \dots, n+1$
is a discrete function on $[0, 1]$



$$D_h := \{(V_0, V_1, \dots, V_{n+1}) \in \mathbb{R}^{n+2}\}$$

= set of discrete functions
on mesh with stepsize h
on $[0, 1]$.

Discrete Poisson eqn = Find $v \in D_h$ such that

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} = f(x_j) \quad j=1, 2, \dots, n$$

$$v_0 = v_{n+1} = 0$$

Equivalently,

$$-v_{j+1} + 2v_j - v_{j-1} = h^2 f(x_j) \quad j=1, 2, \dots, n$$

$$v_0 = v_{n+1} = 0$$

This is a linear system of equations:

$$\begin{array}{rcl} 2v_1 - v_2 & & = h^2 f(x_1) \\ -v_1 + 2v_2 - v_3 & & = h^2 f(x_2) \\ & -v_2 + 2v_3 - v_4 & = h^2 f(x_3) \\ & & \vdots \\ & -v_{n-1} + 2v_n & = h^2 f(x_n) \end{array}$$

Can be written

$$\underbrace{\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & \ddots & \ddots \\ & & & -1 & 2 \end{bmatrix}}_{A \in \mathbb{R}^{n \times n}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_v = \underbrace{\begin{bmatrix} h^2 f(x_1) \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_n) \end{bmatrix}}_b$$

$Av = b$ has unique

solution since A is invertible
(will study next week).