

Lecture 4

Summary Poisson's eq

$$\left. \begin{aligned} -u''(x) &= f(x) \\ u(0) &= u(1) = 0 \end{aligned} \right\} (1)$$

has unique solution

$$u(x) = \int_0^1 G(x, y) f(y) dy.$$

Discrete Poisson problem:

$$\left. \begin{aligned} -\frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} &= f(x_j) \quad j=1, 2, \dots, n \\ v_0 &= v_{n+1} = 0 \end{aligned} \right\} (2)$$

on a given mesh $x_j = jh$, $h = \frac{1}{n+1}$.

We seek a solution v that belongs to $D_{h,0} := \{v \in D_h \mid v_0 = v_{n+1} = 0\}$,

where $D_h = \{(v_0, v_1, \dots, v_{n+1}) \in \mathbb{R}^{n+2}\}$.

Eq (2) can be written

$$\underbrace{\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}}_{A \in \mathbb{R}^{n \times n}} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}}_v = \underbrace{\begin{bmatrix} h^2 f(x_1) \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{n-1}) \\ h^2 f(x_n) \end{bmatrix}}_b$$

For any $n \geq 1$, A is symmetric and invertible, and $v = A^{-1}b$ is unique solution to (2) (on all interior points).
 v_j for $j=1, \dots, n$

Verification

$$\text{Symmetry: } A_{ij} = \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } i=j \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow A_{ij} = A_{ji} \quad \forall i, j \Rightarrow A = A^T.$$

Let $(x, y) := x^T y$ for $x, y \in \mathbb{R}^n$

denote inner product on $\mathbb{R}^n$.

For later reference, note that

$$(Ax, y) = (x, Ay) \quad \forall x, y \in \mathbb{R}^n \quad (3)$$

since $(Ax)^T y = x^T A^T y = x^T (Ay)$.

In fact, A is symmetric $\Leftrightarrow$ (3) holds.

Invertibility of A : For any

$x = (x_1, \dots, x_n) \in \mathbb{R}^n$, introduce $x_0 = x_{n+1} = 0$
and observe that

$$\begin{aligned} (x, Ax) &= \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} -x_0 + 2x_1 - x_2 \\ -x_1 + 2x_2 - x_3 \\ \vdots \\ -x_{n-1} + 2x_n - x_{n+1} \end{bmatrix} \right) \\ &= \sum_{j=1}^n x_j (-x_{j-1} + 2x_j - x_{j+1}) \\ &= \sum_{j=1}^n x_j (x_j - x_{j-1}) - x_j (x_{j+1} - x_j) \\ &= \sum_{j=0}^n x_{j+1} (x_{j+1} - x_j) - x_j (x_{j+1} - x_j) \\ &= \sum_{j=0}^n (x_{j+1} - x_j)^2 \geq 0 \end{aligned} \quad (4)$$

This implies that

$$(x, Ax) = 0 \Leftrightarrow x_j = x_{j+1} \quad j=0, 1, \dots, n$$

$$\Leftrightarrow x = 0 \in \mathbb{R}^n, \text{ since } x_0 = 0.$$

We conclude that A is positive definite:

$$(x, Ax) > 0 \quad \forall x \in \mathbb{R}^n \setminus \{0\},$$

and thus invertible.

Operators L and L_h

Introducing $L: C^2(0,1) \rightarrow C(0,1)$ by

$$(Lu)(x) := -u''(x) \quad x \in (0,1)$$

Then, given $f \in C(0,1)$, eq (1)

can be written:

Find $u \in C_0^2(0,1)$ such that

$$(Lu)(x) = f(x) \quad \forall x \in (0,1).$$

We similarly define

$$L_h: D_h \rightarrow D_{h,0} \quad \text{by}$$

$$\left. \begin{aligned} (L_h v)(x_j) &= - \frac{v(x_{j-1}) - 2v(x_j) + v(x_{j+1}))}{h^2} \\ &\text{for } j=1, 2, \dots, n \\ &\text{and } L_h v(x_0) = L_h v(x_{n+1}) = 0 \end{aligned} \right\} (5)$$

For $v \in C_0([0,1])$, we also define $(L_h v)(x_j)$ by (5).

Note: For $v \in D_{h,0}$ with $\bar{v} := (v_1, \dots, v_n)$,

$$L_h v = \begin{bmatrix} 0 \\ h^{-2} A \bar{v} \\ 0 \end{bmatrix}, \quad (6)$$

Discrete Poisson's eq: Find $v \in D_{h,0}$ s.t.

$$(L_h v)(x_j) = f(x_j) \quad \text{for all } j=1, 2, \dots, n.$$

Inner products

For $u, v \in C([0,1])$

$$\langle u, v \rangle := \int_0^1 u(x) v(x) dx$$

and for $u, v \in D_h$

$$\langle u, v \rangle_h := h \left(\frac{u_0 v_0 + u_{n+1} v_{n+1}}{2} + \sum_{j=1}^n u_j v_j \right)$$

Lemma (symmetries of L & L_h)

①

$$\langle Lu, v \rangle = \langle u, Lv \rangle \quad \forall u, v \in C_0^2([0,1])$$

and

$$\textcircled{2} \quad \langle L_h u, v \rangle_h = \langle u, L_h v \rangle_h \quad \forall u, v \in P_{h,0}$$

Proof

$$\begin{aligned} \textcircled{1} \quad \int_0^1 -u''(x) v(x) dx &= -[u' v]_0^1 + \int_0^1 u' v' dx \\ &= [u v']_0^1 - \int_0^1 u v'' dx = \int_0^1 u(x) (-v'')(x) dx \end{aligned}$$

(2)

Writing $u = \begin{bmatrix} 0 \\ \bar{u}_1 \\ \vdots \\ \bar{u}_n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{u} \\ 0 \end{bmatrix}$

and similarly $v = \begin{bmatrix} 0 \\ \bar{v} \\ 0 \end{bmatrix}$, we obtain

$$\langle L_h u, v \rangle_h \stackrel{(6)}{=} \left\langle \begin{bmatrix} 0 \\ h^{-2} A \bar{u} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{v} \\ 0 \end{bmatrix} \right\rangle_h$$

$$= \frac{h}{h^2} \sum_{j=1}^n (A \bar{u})_j v_j \quad ,$$

$$= \frac{1}{h} (A \bar{u}, \bar{v})$$

$$\stackrel{(3)}{=} \frac{1}{h} (\bar{u}, A \bar{v})$$

$$= \langle u, L_h v \rangle_h$$

Lemma (Positive definiteness of L & L_h)

① For any $u \in C_0^2(0,1)$,

$$\langle Lu, u \rangle \geq 0 \quad \text{and}$$

$$\langle Lu, u \rangle = 0 \Rightarrow u \equiv 0.$$

② For any $v \in D_{h,0}$,

$$\langle L_h v, v \rangle_h \geq 0 \quad \text{and}$$

$$\langle L_h v, v \rangle_h = 0 \Rightarrow v \equiv 0$$

Proof

① We have seen that

$$\langle Lu, u \rangle = - \int_0^1 u(x) u''(x) dx$$

$$= \underbrace{[u u']_0^1}_{=0} + \int_0^1 (u'(x))^2 dx \geq 0.$$

And

$$\langle Lu, u \rangle = 0 \Rightarrow u' \equiv 0$$

$\Rightarrow u$ is constant

$$\Rightarrow u \equiv 0 \quad \text{since } u(0) = 0.$$

$$\textcircled{2} \quad \langle L_h v, v \rangle_h = \left\langle \begin{bmatrix} 0 \\ h^2 A \bar{v} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \bar{v} \\ 0 \end{bmatrix} \right\rangle_h$$

$$= \frac{1}{h} (A \bar{v}, \bar{v}).$$

result follows from A being Positive definite.

Lemma (Uniqueness)

If $f \in C([0,1])$, then the solution to (1) and (2) are both unique

Proof for (1): Let $u, \hat{u} \in C_0^2(0,1)$ both be solutions to (1), meaning

$$Lu = f \quad \text{and} \quad L\hat{u} = f.$$

Let $e = u - \hat{u}$. Since L is linear

$$Le = L(u - \hat{u}) = Lu - L\hat{u} = 0$$

$$\Rightarrow e(x)(Le)(x) = 0 \quad \forall x \in (0,1)$$

$$\Rightarrow \langle Le, e \rangle = \int_0^1 (Le)(x) e(x) dx = 0$$

$$e \in C_0^2([0,1])$$

$$\Rightarrow e \equiv 0 \Rightarrow u \equiv \hat{u}.$$

Alternative representation of discrete solution

$$\text{Using } G(x,y) = \begin{cases} y(1-x) & y \leq x \\ x(1-y) & x \leq y \end{cases}$$

we will show that solution of (2) can be written

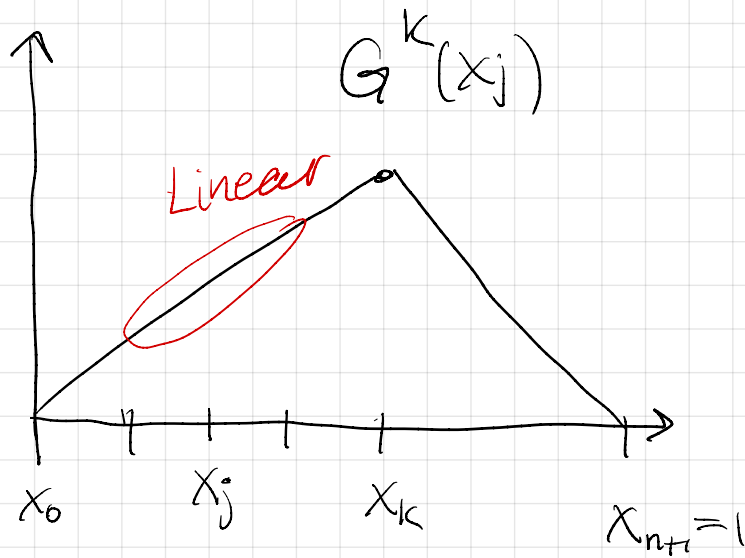
$$v(x_j) = h \sum_{k=1}^n G(x_j, x_k) f(x_k) \quad 1 \leq j \leq n.$$

compare this to solution of (1):

$$u(x) = \int_0^1 G(x,y) f(y) dy.$$

Verification For any $1 \leq k \leq n$, let $G^k(\cdot) \in D_{n,0}$ be defined by

$$G^k(x_j) = G(x_j, x_k).$$



Then for $j \neq k$, $1 \leq j \leq n$

$$(L_h G^k)(x_j) = 0 \quad \text{since } G^k|_{[x-h, x+h]} \text{ is linear}$$

while

$$(L_h G^k)(x_k) = \frac{-(x_k-h)(1-x_k) + 2x_k(1-x_k) - x_k(1-x_k-h)}{h^2}$$

$$= \frac{1}{h}$$

Defining basis functions $e^k \in D_{n,0}$ by

$$e^k(x_j) = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{otherwise,} \end{cases}$$

yields that

$$L_h G^k = \frac{e^k}{h}.$$

For any $f \in D_n$, define $v \in D_{n,0}$ by

$$v(x_j) := h \sum_{k=1}^n f(x_k) G^k(x_j).$$

Then

$$\begin{aligned} (L_h v)(x_j) &\stackrel{\substack{\text{L}_h \text{ linear}}}{=} h \sum_{k=1}^n f(x_k) (L_h G^k)(x_j) \\ &= \sum_{k=1}^n f(x_k) e^k(x_j) \\ &= f(x_j) \quad \text{for all } j=1,2,\dots,n \end{aligned}$$

Conclusion: v is the solution to (2).