

Lecture 5

Poisson's eq

Find $u \in C_0^2([0,1])$ s.t. (1)
 $-u''(x) = (Lu)(x) = f(x)$

Discrete eq.

Find $v \in D_{h,0}$ s.t. (2)
 $(L_h v)(x_j) = f(x_j)$
for $j=1,2,\dots,n$

Solutions:

$$u(x) = \int_0^1 G(x,y) f(y) dy,$$

$$v(x_j) = h \sum_{k=1}^n G(x_j, x_k) f(x_k)$$

Example

when $f \equiv 1$, then solution of (2) is

$$V(x_j) = \frac{1}{2} x_j(1-x_j) \quad (\text{Exercise 2.16})$$

$$\Rightarrow h \sum_{k=1}^n G(x_j, x_k) = \frac{1}{2} x_j(1-x_j) \quad (3)$$

Maximum principle

For $g \in D_n$, let $\|g\|_{n,\infty} := \max_{j=0,1,\dots,n} |g(x_j)|$,
and let $v \in D_{n,0}$ be solution of (2).

① If $f \geq 0$, then $v(x_j) \geq 0$ for all
 $j = 1, 2, \dots, n$

$$\textcircled{2} \quad \|v\|_{n,\infty} \leq \frac{\|f\|_{n,\infty}}{8}$$

Proof

$$\textcircled{1} \quad v(x_j) = h \sum_{k=1}^n \underbrace{G(x_j, x_k)}_{\geq 0} \underbrace{f(x_k)}_{\geq 0} \geq 0$$

$$\textcircled{2} \quad |V(x_j)| \leq h \sum_{k=1}^n G(x_j, x_k) |f(x_k)|$$

$$\leq \|f\|_{h,\infty} \left(h \sum_{k=1}^n G(x_j, x_k) \right)$$

$$\textcircled{3} \quad \|f\|_{h,\infty} \frac{x_j(1-x_j)}{2} \leq \frac{\|f\|_{h,\infty}}{8}$$

Convergence of solutions

Let u be solution of (1) and v solution of (2).

The vector $\tau_h \in D_{n,0}$ defined by

$$\tau_h(x_j) = (L_h u)(x_j) - f(x_j) \quad j=1, \dots, n$$

is called the truncation error.

Properties:

① If $f \in C^2([0,1])$, then

$$\|\tau_h\|_{h,\infty} \leq \frac{\|f''\|_{h,\infty}}{12} h^2$$

② Convergence rate:

$$\|u - v\|_{h,\infty} \leq \frac{\|f''\|_{h,\infty}}{96} h^2$$

Proof of ②

Define error function $e \in D_{h,0}$ by

$$e(x_j) = u(x_j) - v(x_j) \quad j=1, \dots, n.$$

Then

$$(L_h e)(x_j) = (L_h u)(x_j) - (L_h v)(x_j)$$

$$= (L_h u)(x_j) - f(x_j)$$

$$= \tau_h(x_j)$$

$\Rightarrow e$ is solution to (2) with $f = \tau_h$

maximum principle

$$\Rightarrow \|e\|_{h,\infty} \leq \frac{\|\tau_h\|_{h,\infty} \textcircled{1}}{8} \leq \frac{\|f\|_{h,\infty}}{96} h^2.$$

Eigenvalue problems

A number $\lambda \in \mathbb{R}$ is an eigenvalue associated to problem (1) if

$$Lu = \lambda u \quad (4)$$

for some nonzero $u \in C_0^2([0,1])$.
 u is then called an eigenfunction

(Extension of linear algebra =

$$Ax = \lambda x, \quad \lambda\text{-eigenvalue,} \\ x\text{-eigenvector.})$$

Plan: Find all eigenpairs (λ, u) to (1)

Recall that

$$\langle Lu, u \rangle > 0 \quad \forall u \in C_0^2([0,1])$$

Suppose that (λ, u) is eigenpair in (4).
Then

$$0 < \langle Lu, u \rangle = \lambda \langle u, u \rangle$$

So must have $\lambda > 0$.

We define $\beta = \sqrt{\lambda}$ and rewrite (4):

$$u''(x) + \beta u(x) = 0,$$

$$u(0) = u(1) = 0$$

General solution

$$u(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x)$$

$$u(0) = 0 \Rightarrow C_1 = 0$$

$$\text{and } u(1) = 0 \Rightarrow C_2 \sin(\beta) = 0$$

$$\Rightarrow \beta = k\pi, \quad k = 1, 2, \dots$$

The eigenpairs (λ_k, u_k) of (4) are given by

$$\lambda_k = (k\pi)^2 \quad \text{and } u_k(x) = \sin(k\pi x)$$

for $k = 1, 2, \dots$

Lemma (orthogonality of eigenfunctions)

$$\langle \sin(k\pi x), \sin(m\pi x) \rangle = \begin{cases} 0 & k \neq m \\ 1/2 & k = m \end{cases}$$

Proof

Use that $\sin(a)\sin(b) = \frac{\cos(a-b) - \cos(a+b)}{2}$

$$\int_0^1 \sin(k\pi x) \sin(m\pi x) dx$$

$$= \frac{1}{2} \int_0^1 (\cos((k-m)\pi x) - \cos((k+m)\pi x)) dx$$

$$= \begin{cases} \frac{1}{2} \left(1 - \left[\frac{\sin((k+m)\pi x)}{(k+m)\pi} \right]_0^1 \right) & k=m \\ \frac{1}{2} \left[\frac{\sin((k-m)\pi x)}{(k-m)\pi} + \frac{\sin((k+m)\pi x)}{(k+m)\pi} \right]_0^1 & k \neq m \end{cases}$$

We also consider the discrete eigenvalue problem:

Find $\mu \in \mathbb{R}$ and $v \in D_{h,0}$ such that

$$L_h v = \mu v \quad (5)$$

How many eigenpairs?

Realizing that

$$L_h v = \begin{bmatrix} 0 \\ h^{-2} A \bar{v} \\ 0 \end{bmatrix} \quad \text{where } v = \begin{bmatrix} 0 \\ \bar{v} \\ 0 \end{bmatrix}$$

$$\text{and } A = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

(5) is equivalent to: find $\mu \in \mathbb{R}$ and $\bar{v} \in \mathbb{R}^n$ s.t.

$$h^{-2} A \bar{v} = \mu \bar{v}.$$

So (5) has n eigenpairs and all eigenvalues are positive since A is positive definite.

Using that

$$\sin(a-b) + \sin(a+b) = 2\cos(b)\sin(a)$$

and eigenfunction guess

$$V_k(x_j) := \sin(k\pi x_j)$$

one obtains that

$$L_h V_k(x_j) = h^{-2} (-\sin(k\pi(x_j-h)) + 2\sin(k\pi x_j) - \sin(k\pi(x_j+h)))$$

$$= \underbrace{\frac{2}{h^2} (1 - \cos(k\pi h))}_{\mu_k} \sin(k\pi x_j)$$

and $V_k \in D_{h,0}$.

The eigenpairs of (5) are given by

$$\mu_k = \frac{2}{h^2} (1 - \cos(k\pi h)) = \frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$$

and

$$V_k(x_j) = \sin(k\pi x_j)$$

for $k = 1, 2, \dots, n$

Moreover,

$$\langle V_k, V_m \rangle_h = h \sum_{j=1}^n V_k(x_j) V_m(x_j) = \begin{cases} 0 & k \neq m \\ 1/2 & k = m \end{cases} \quad (6)$$

(see exercise 2.30),

$\{V_k\}_{k=1}^n$ is thus an orthogonal basis

for $D_{n,0}$: any $g \in D_{n,0}$ can be written

$$g = \sum_{k=1}^n c_k V_k$$

Where coefficients $c_1, \dots, c_n \in \mathbb{R}$ are given by

$$\langle g, V_m \rangle_h = \sum_{k=1}^n c_k \langle V_k, V_m \rangle \stackrel{(6)}{=} \frac{c_m}{2}$$

$$\Rightarrow c_m = 2 \langle g, V_m \rangle_h \quad m=1, \dots, n$$

and

$$g(x_j) = \sum_{k=1}^n \underbrace{2 \langle g, V_k \rangle_h}_{c_k} \underbrace{\sin(k\pi x_j)}_{V_k(x_j)} \quad (7)$$

Formal limits as $n \rightarrow \infty$ and $h \rightarrow 0$:

$$\langle \cdot, \cdot \rangle_h \longrightarrow \langle \cdot, \cdot \rangle$$

$$D_{n,0} \longrightarrow C_0^2([0,1])$$

discrete problem

continuous problem

Question: Does (7) extend as $h \rightarrow 0$

such that functions $f: [0,1] \rightarrow \mathbb{R}$ can be written

$$f(x) = \sum_{k=1}^{\infty} 2 \langle f, \sin(k\pi x) \rangle \sin(k\pi x) \quad ?$$