

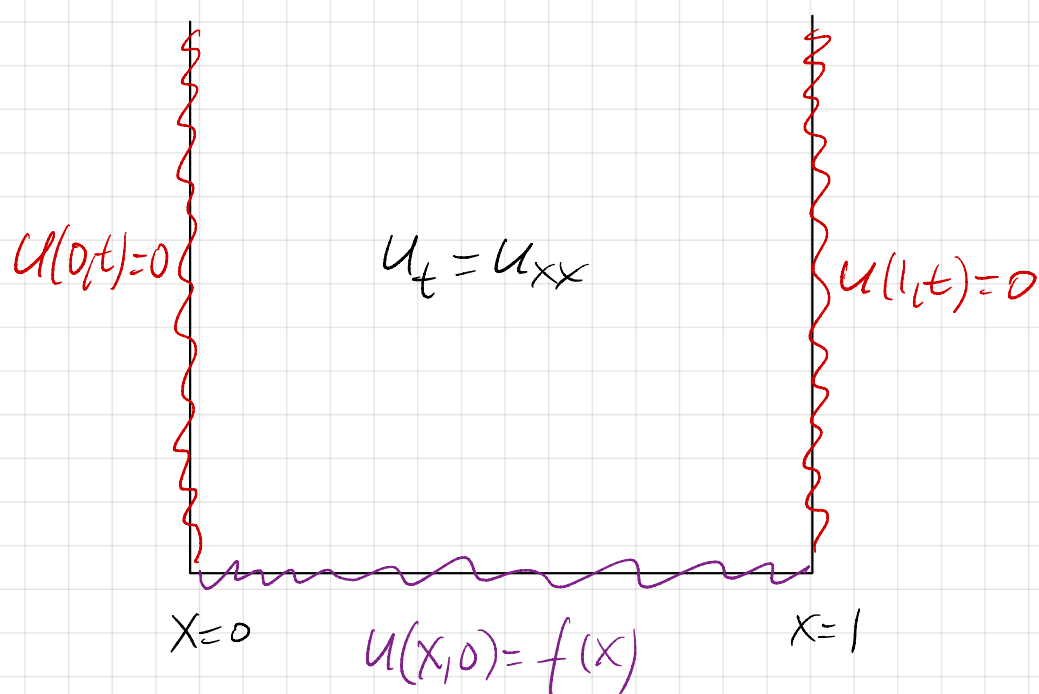
Lecture 6

Heat eq on bounded domain

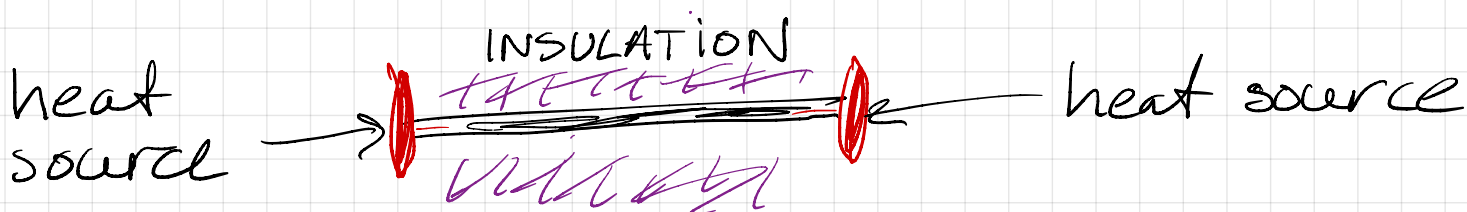
$$\left. \begin{aligned} u_t &= u_{xx} & x \in (0,1), t > 0 \\ u(0,t) &= u(1,t) = 0 & t \geq 0 \end{aligned} \right\} (1)$$

and with initial condition

$$u(x,0) = f(x) \quad x \in (0,1) \quad (2)$$



Models evolution of heat in uniform rod with "heat" sources at end points.



Fourier's method

① Find family of solutions $\{u_k(x,t)\}$ to (1)

② Observe that any linear combination
$$u(x,t) = \sum_{k=1}^N c_k u_k(x,t) \quad , \quad \begin{matrix} N < \infty \\ c_k \in \mathbb{R} \end{matrix}$$
also solves (1) (since it is a linear PDE).

③ Find coefficients $\{c_k\}$ s.t.

$$f(x) = \sum_k c_k u_k(x,0).$$

① Separation of variables

We seek solution $u_k(x,t)$

$$(u_k)_t = (u_k)_{xx} \quad x \in (0,1), \quad t > 0 \quad (3)$$

$$u_k(0,t) = u_k(1,t) = 0 \quad t > 0$$

Make the guess that

$$u_k(x,t) = X_k(x) T_k(t),$$

for some $X_k \in C_0^2((0,1))$ and $T_k \in C^1([0,\infty))$

Inserted into (3) yields

$$\Sigma_k(x) T_k'(t) = \Sigma_k''(x) T_k(t) \quad | \cdot \frac{1}{\Sigma_k T_k}$$

$$\Rightarrow \underbrace{\frac{T_k'(t)}{T_k(t)}}_{\text{only depends on } t} = \underbrace{\frac{\Sigma_k''(x)}{\Sigma_k(x)}}_{\text{only depends on } x}$$

$$\Rightarrow \frac{T_k'(t)}{T_k(t)} = \frac{\Sigma_k''(x)}{\Sigma_k(x)} = -\lambda_k \quad (4)$$

for some constant $\lambda_k \in \mathbb{R}$.

Rewrite (4) as

$$\left. \begin{aligned} -\Sigma_k''(x) &= \lambda_k \Sigma_k(x) \quad x \in (0,1) \\ \Sigma_k(0) &= \Sigma_k(1) = 0 \end{aligned} \right\} (5)$$

and

$$T_k'(t) + \lambda_k T_k(t) = 0 \quad t > 0$$

$$T_k(0) = 1$$

(any $T_k(0) \neq 0$ boundary condition would work).

(5) is the eigenvalue problem

$$(L X_k)(x) = \lambda_k X_k(x)$$

with eigenpairs

$$X_k(x) = \sin(k\pi x) \quad \text{and} \quad \lambda_k = (k\pi)^2 \quad k=1,2,\dots$$

and

$$T_k(t) = e^{-\lambda_k t} T_k(0) = e^{-(k\pi)^2 t}$$

Family of solutions:

$$u_k(x,t) = X_k(x) T_k(t) = e^{-(k\pi)^2 t} \sin(k\pi x)$$

for $k=1,2,\dots$

② Principle of superposition

For linear homogeneous PDE

$$u_t = u_{xx}$$

with 0-valued boundary cond

$$u(0, t) = u(1, t) = 0,$$

a linear combination of solutions is a solution:

$$u(x, t) = \sum_{k=1}^N c_k u_k(x, t) \quad , \quad \begin{matrix} N < \infty \\ c_k \in \mathbb{R} \end{matrix}$$

(Exercise: verify).

What is $u(x, 0)$?

$$u(x, t) = \sum_{k=1}^N c_k e^{-(k\pi)^2 t} \sin(k\pi x) ,$$

$$\text{so } u(x, 0) = \sum_{k=1}^N c_k \sin(k\pi x) =: f(x).$$

We can now solve (1)-(2) for some initial conditions:

Ex For $f(x) = \sin(\pi x) + 3\sin(3\pi x) + 7\sin(5\pi x)$

Solution to (1)-(2) is

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x) + 3e^{-(3\pi)^2 t} \sin(3\pi x) + 7e^{-(5\pi)^2 t} \sin(5\pi x)$$

Summary: Consider

$$\left. \begin{aligned} u_t &= u_{xx} \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= f(x) \end{aligned} \right\} (6)$$

if $f(x) = \sum_{k=1}^N c_k \sin(k\pi x)$

then $u(x,t) = \sum_{k=1}^N c_k e^{-(k\pi)^2 t} \sin(k\pi x)$

is a solution.

And in formal limit $N \rightarrow \infty$,

$$f(x) = \sum_{k=1}^{\infty} C_k \sin(k\pi x)$$

then

$$u(x, t) = \sum_{k=1}^{\infty} C_k e^{-(k\pi)^2 t} \sin(k\pi x) .$$

To solve (6) for given $f(x)$, we need to find coefficients $\{C_k\}$ s.t.

$$\sum_{k=1}^{\infty} C_k \sin(k\pi x) = f(x) \quad (7)$$

Recall that $v_k(x) := \sin(k\pi x)$ are orthogonal:

$$\langle v_k, v_m \rangle = \int_0^1 v_k(x) v_m(x) dx = \begin{cases} 0 & m \neq k \\ 1/2 & m = k \end{cases} \quad (8)$$

Rewrite (7) as

$$\sum_{k=1}^{\infty} C_k v_k(x) = f(x)$$

$$| \langle \cdot, v_m \rangle$$

and determine c_m by

$$\langle f, v_m \rangle = \sum_{k=1}^{\infty} c_k \langle v_k, v_m \rangle$$

$$\stackrel{(8)}{=} \frac{c_m}{2}$$

$$\Rightarrow c_m = 2 \langle f, v_m \rangle$$

and

$$f(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

Example

The PDE (6) with $f(x) = 1 \quad x \in [0,1]$
Fourier coefficients

$$c_k = 2 \langle f, v_k \rangle$$

$$= 2 \int_0^1 \sin(k\pi x) dx$$

$$= \frac{2(1 - \cos(k\pi))}{k\pi} = \begin{cases} \frac{4}{k\pi} & k=1, 3, 5, \dots \\ 0 & k=2, 4, 6, \dots \end{cases}$$

Fourier sine series:

$$f(x)=1 = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin((2k-1)\pi x)$$

Note: This is a formal equality that holds for all $x \in (0,1)$, but for

$$x=0$$

$$f(0)=1 \neq \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \underbrace{\sin((2k-1)\pi \cdot 0)}_{=0} = 0$$

Formal solution to PDE:

$$u(x,t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{e^{-(2k-1)^2 \pi^2 t}}{2k-1} \sin((2k-1)\pi x)$$

Ex For $f(x) = x$, the Fourier coefficients are

$$C_k = 2 \langle f, v_k \rangle$$

$$= 2 \int_0^1 x \sin(k\pi x) dx$$

$$= 2 \left[\frac{-x \cos(k\pi x)}{k\pi} \right]_0^1$$

$$+ 2 \underbrace{\int_0^1 \frac{\cos(k\pi x)}{k\pi} dx}_0$$

$$= -2 \frac{\cos(k\pi)}{k\pi} = (-1)^{k+1} \frac{2}{k\pi}$$

and sine series

$$x = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin(k\pi x)$$

Types of boundary data:

Dirichlet: $u(0,t) = a$, $u(l,t) = b$

Neumann: $u_x(0,t) = a$, $u_x(l,t) = b$

Robin: $\alpha u_x(0,t) + \beta u(0,t) = a$
 $\gamma u_x(l,t) + \delta u(l,t) = b$

For some $\alpha, \beta, \gamma, \delta, a, b \in \mathbb{R}$.

Periodic: $u(0,t) = u(l,t)$, $u_x(0,t) = u_x(l,t)$

Next time: Fourier's method
for heat eq with Neumann
boundary conditions.